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PRACTICE PAPER 07 (2023-24)

CHAPTER 07 INTEGRALS (ANSWERS)

SUBJECT: MATHEMATICS **MAX. MARKS : 40**

CLASS : XII **DURATION : $1\frac{1}{2}$ hrs**

General Instructions:

(i). **All** questions are compulsory.

(ii). This question paper contains 2**0** questions divided into five Sections A, B, C, D and E.

(iii). **Section A** comprises of **10 MCQs** of **1 mark** each. **Section B** comprises of 4 questions of **2 marks** each. **Section C** comprises of 3 questions of **3 marks** each. **Section D** comprises of 1 question of **5 marks** each and **Section E** comprises of 2 Case Study Based Questions of **4 marks** each.

(iv). There is no overall choice.

(v). Use of Calculators is not permitted

SECTION – A

Questions 1 to 10 carry 1 mark each.

1. The value of $\int_0^{2\pi} \frac{dx}{e^{\sin x}+1}$ is

(a) π (b) 0 (c) 3 π (d) π/2

Ans: (a) π

Let $I = \int_0^{2\pi} \frac{dx}{e^{\sin x}+1}$ ...(i)

$\Rightarrow I = \int_0^{2\pi} \frac{dx}{e^{\sin(2\pi - x)}+1}$ $\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{2\pi} \frac{dx}{e^{-\sin x}+1}$ $\Rightarrow I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}dx$

Adding (i) and (ii), we get

$2I = \int^{2\pi} 1 \cdot dx = 2\pi \quad \therefore \quad I = \pi$

2. Evaluate: $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

(a) tanx – cotx + C (b) –tanx + cotx + C

(c) tanx + cotx + C (d) –tanx – cotx +C

Ans: (c) tanx + cotx + C

3. The value of is $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$:

(a) π/3 (b) π/2 (c) π/4 (d) π/6

Ans: (a) π/3

Let $I = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_1^2 = \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

4. The value of $\int_0^a \frac{\sqrt{a}}{\sqrt{x}+\sqrt{a-x}} dx$ is:

(a) a/2 (b) a (c) a^2 (d) 0

Ans: (a) a/2

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(i)$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}dx$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(ii)$$

Adding (i) and (ii), we get $2I = \int_0^a 1.dx \Rightarrow 2I = a$ or $I = \frac{a}{2}$

5. The value of $\int \frac{1}{e^x - 1}dx$ is:

(a) $\log e^x + C$ (b) $\log|1 - e^{-x}| + C$

(c) $\log\log\frac{1}{e^x} + C$ (d) $\log|e^x - 1| + C$

Ans: (b) $\log|1 - e^{-x}| + C$

Let $I = \int \frac{1}{e^x - 1}dx.$

Put $e^x - 1 = t$ so that $e^x\,dx = dt$

$$= \int\left(\frac{1}{t}\cdot\frac{1}{t+1}\right)dt = \int\frac{1}{t}dt - \int\frac{1}{t+1}dt$$

$$= \log|t| - \log|t+1| + c = \log|e^x - 1| - \log|e^x| + c$$

$$= \log|1 - e^{-x}| + c$$

6. The value of $\int_{-1}^{1}(x-[x])dx$ is:

(a) -1 (b) 0 (c) 1 (d) 2

Ans: (c) 1

$$I = \int_{-1}^{1}(x-[x])dx = \int_{-1}^{1}xdx - \int_{-1}^{1}[x]dx = 0-(-1)\int_{-1}^{0}dx - 0\int_{0}^{1}dx \quad [\because \text{ x is an odd function}]$$

$= 0 + 1 - 0 = 1$

7. The value of $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ is:

(a) $\pi/3$ (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

Ans: (a) $\pi/3$

Let $I = \int_1^2 \frac{dx}{x\sqrt{x^2-1}}$

$$= \left[\sec^{-1}x\right]_1^2 = \sec^{-1}2 - \sec^{-1}1 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

8. $\int \cos^3 x.e^{\log(\sin x)}dx$ is equal to

(a) $-\frac{\cos^4 x}{4}+C$ (b) $-\frac{\sin^4 x}{4}+C$ (c) $\frac{e^{\sin x}}{4}+C$ (d) none of these

Ans: (a) $-\frac{\cos^4 x}{4}+C$

In the following questions 9 and 10, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:

(a) Both Assertion (A) and Reason (R) are true and Reason(R) is the correct explanation of assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason(R) is not the correct explanation of assertion (A).
(c) Assertion (A) is true but reason (R) is false.
(d) Assertion (A) is false but reason (R) is true.

9. Assertion(A): $\int \frac{1}{\sqrt{x^2+2x+10}}dx = \sin^{-1}\left(\frac{x+1}{3}\right)+C$

Reason(R) : If a > 0, $b^2 - 4ac < 0$ then $\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C$

Ans: (a) Both A and R are true and R is the correct explanation of A.

10. Assertion(A): $\int_{-3}^{3}(x^3+5)dx = 30$

Reason(R): $f(x) = x^3 + 5$ is an odd function.
Ans. (c) A is true but R is false.
Let $f(x) = x^3 + 5$
$f(-x) = (-x)^3 + 5 = -x^3 + 5$
f(x) is neither even nor odd.
Hence R is false.
$\int_{-3}^{3}(x^3+5)dx = \int_{-3}^{3}x^3dx + \int_{-3}^{3}5dx = 0 + 5[x]_{-3}^{3} = 30$
Hence A is true.

SECTION – B

Questions 11 to 14 carry 2 marks each.

11. Evaluate: $\int \sqrt{1-\sin 2x}\,dx,\ \frac{\pi}{4} < x < \frac{\pi}{2}$

Ans:
Let $I = \int (\sqrt{1-\sin 2x})\,dx$
$= \int \sqrt{\cos^2 x + \sin^2 x - 2\sin x\cos x}\,dx$
$= \pm \int (\cos x - \sin x)\,dx$
Since, $\frac{\pi}{4} < x < \frac{\pi}{2}$, so we get $I = \int (\sin x - \cos x)\,dx = -(\cos x + \sin x) + C$

12. Find the value of $\int_{1}^{2} \frac{dx}{x(1+\log x)^2}$.

Ans:
Let $I = \int_{1}^{2} \frac{dx}{x(1+\log x)^2}$
Put $1 + \log x = t \Rightarrow \frac{dx}{x} = dt$
When $x = 1$, $t = 1$ and when $x = 2$, $t = 1 + \log 2$
$\therefore\quad I = \int_{1}^{1+\log 2} \frac{dt}{t^2} = \left[\frac{-1}{t}\right]_{1}^{1+\log 2} = -\left[\frac{1}{1+\log 2} - 1\right] = -\left[\frac{1-1-\log 2}{1+\log 2}\right] = \frac{\log 2}{1+\log 2}$

13. Evaluate: $\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

$$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \quad ...(i)$$

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad ...(ii)$$

Adding (i) and (ii)

$$2I=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$

Putting $\cos x=t$ gives $I=\frac{\pi}{2}\int_{-1}^{1}\frac{dt}{1+t^2}$

$$\Rightarrow I=\frac{\pi}{2}\left[\tan^{-1}t\right]_{-1}^{1}=\frac{\pi^2}{4}$$

14. Evaluate: $\int\frac{dx}{9x^2+6x+10}$.

Let $I=\int\frac{1}{9x^2+6x+10}dx=\frac{1}{9}\int\frac{1}{x^2+\frac{2}{3}x+\frac{10}{9}}dx$

$$=\frac{1}{9}\int\frac{1}{x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}+\frac{10}{9}}dx$$

$$=\frac{1}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+(1)^2}dx=\frac{1}{9}\times\frac{1}{1}\tan^{-1}\left(\frac{x+\frac{1}{3}}{1}\right)+C$$

$$=\frac{1}{9}\tan^{-1}\left(\frac{3x+1}{3}\right)+C$$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Evaluate: $\int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}$

Ans:

Let $I=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad ...(i)$

By the property, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$$I=\int_0^{\pi/2}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}+\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right]dx$$

$$=\int_0^{\pi/2}1\cdot dx=[x]_0^{\pi/2}=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

16. Evaluate: $\int e^x \frac{(1-\sin x)}{(1-\cos x)} dx$

Ans:

$$\int e^x \frac{(1-\sin x)}{(1-\cos x)} dx = \int e^x \frac{\left(1-2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{2\sin^2\frac{x}{2}} dx$$

$$= \int e^x \left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$$

$$= \int e^x \left[\frac{1}{2}\text{cosec}^2\frac{x}{2} + \left(-\cot\frac{x}{2}\right)\right] dx \qquad \left[\because \text{ where } f(x) = -\cot\frac{x}{2},\ f'(x) = \frac{1}{2}\text{cosec}^2\frac{x}{2}dx\right.$$

$$= \int e^x [f'(x) + f(x)]\, dx$$

$$= e^x f(x) + c = -e^x . \cot\frac{x}{2} + c$$

17. Evaluate: $\int \frac{3x+1}{(x-1)^2(x+3)} dx$

Ans:

Using partial fraction

$$\frac{3x+1}{(x-1)^2(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)} \quad ...(i)$$

On comparing $3x + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$

Putting $x = 1$, we get $3 \times (1) + 1 = 0 + B(1+3) + 0$

$\Rightarrow 4 = 4B \Rightarrow B = 1$

Putting $x = -3$, we get

$3 \times (-3) + 1 = A(-3-1)(-3+3) + B(-3+3) + C(-3-1)2$

$\Rightarrow -8 = 16C \Rightarrow C = -\frac{1}{2}$

We have $-3A + C = 1 \Rightarrow -3A - \frac{1}{2} = 1$

$\Rightarrow -3A = \frac{3}{2} \Rightarrow A = \frac{1}{2}$

Equation (i) can be written as

$$\frac{3x+1}{(x+3)(x-1)^2} = \frac{1}{2(x-1)} + \frac{1}{(x-1)^2} - \frac{1}{2(x+3)}$$

$$\Rightarrow I = \int \frac{3x+1}{(x+3)(x-1)^2} dx = \frac{1}{2}\int \frac{1}{(x-1)} dx + \int \frac{1}{(x-1)^2} dx - \frac{1}{2}\int \frac{1}{(x+3)} dx$$

$$\Rightarrow I = \frac{1}{2}\log|x-1| - \frac{1}{(x-1)} - \frac{1}{2}\log|x+3| + C$$

SECTION – D

Questions 18 carry 5 marks.

18. Evaluate: $\int \frac{x^2}{x^4 - x^2 - 12} dx$

Ans:

Let $I = \int \frac{x^2}{x^4 - x^2 - 12} dx = \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx$

$= \int \frac{x^2 dx}{x^2(x^2 - 4) + 3(x^2 - 4)} = \int \frac{x^2}{(x^2 - 4)(x^2 + 3)} dx$

$\Rightarrow \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$ [let $x^2 = t$] $\Rightarrow t = A(t+3) + B(t-4)$

On comparing the coefficient of t on both sides, we get $A + B = 1$

$\Rightarrow 3A - 4B = 0 \quad \Rightarrow 3(1 - B) - 4B = 0$

$\Rightarrow 3 - 3B - 4B = 0 \Rightarrow 7B = 3 \Rightarrow B = \frac{3}{7}$

If $B = \frac{3}{7}$, then $A + \frac{3}{7} = 1 \Rightarrow A = 1 - \frac{3}{7} = \frac{4}{7}$

Now, $\frac{x^2}{(x^2-4)(x^2+3)} = \frac{4}{7(x^2-4)} + \frac{3}{7(x^2+3)} = \frac{4}{7}\int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7}\int \frac{1}{x^2 + (\sqrt{3})^2} dx$

$= \frac{4}{7} \cdot \frac{1}{2.2} \log\left|\frac{x-2}{x+2}\right| + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + C = \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. Case-Study 1: Read the following passage and answer the questions given below.

A general election of Lok Sabha is a gigantic exercise. About 911 million people were eligible to vote and voter turnout was about 67%, the highest ever.

Let f(x) be the set of all citizens of India who were eligible to exercise their voting right in the general election held in 2019. A relation 'R' is defined on I as follows:

If f(x) is a continuous function defined on [a, b] $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ on the basis of the above information answer the following equations:

(a) Evaluate: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x} dx$ [2]

(b) Find the value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$, a > 0. [2]

Ans: (a)

Let $\quad I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x} dx \quad$ ...(i)

Applying property, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-\frac{\pi}{2}-x\right)}{1+e^{\left(\frac{\pi}{2}-\frac{\pi}{2}-x\right)}} dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(-x)}{1+e^{-x}} dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \cos x}{1+e^x} dx \quad ...(ii)$$

on adding equation (i) and (ii), we get

$$2I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{(1+e^x)\cos x}{1+e^x} dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$$

Now, cosx is an even function

$$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \cos x \, dx \Rightarrow 2I = 2[\sin x]_0^{\frac{\pi}{2}} = 2(1-0) = 2 \Rightarrow I = 1$$

(b)

Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \quad$ ...(i)

Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(-\pi+\pi-x)}{1+a^{(-\pi+\pi-x)}} dx = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad ...(ii)$$

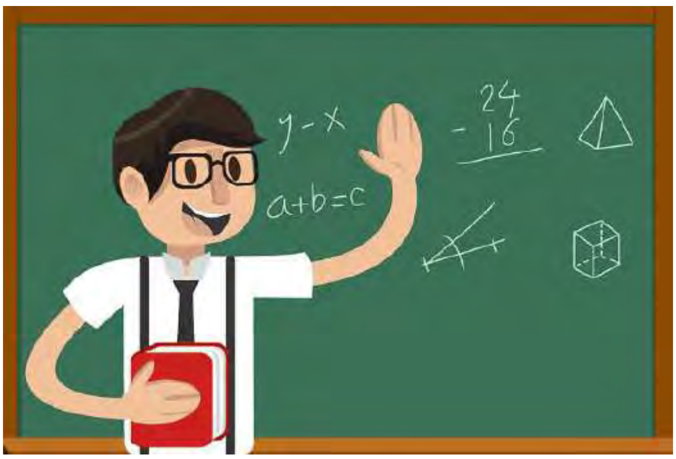
on adding (i) and (ii), we get

$$2I = \int_{-\pi}^{\pi} \cos^2 x \, dx = \left[\frac{1}{2}(x+\sin x \cos x)\right]_{-\pi}^{\pi} = \frac{1}{2}(\pi+0+\pi-0) = \pi$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

20. Case-Study 2:

Mr. Kumar is a Maths teacher. One day he taught students that the Integral $I = \int f(x)dx$ can be transformed into another form by changing the independent variable x to t by substituting



Consider $I = \int f(x)dx$

Put $x = g(t)$ so that $\frac{dx}{dt} = g'(t)$ then we write $dx = g'(t)dt$

Thus, $I = \int f(x)dx = \int f(g(t))g'(t)dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution.

Based on the above information, answer the following questions:

(i) Evaluate: $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$ (2)

(ii) Evaluate: $\int \frac{x^3}{(x^2+1)^3} dx$ (2)

OR

(ii) Evaluate: $\int \frac{dx}{x\sqrt{x^6-1}} dx$ (2)

Ans: (i)

Put $\tan^{-1} x = t$ so that $\frac{1}{(1+x^2)} dx = dt$.

$\therefore \int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx = \int e^t dt = e^t + C = e^{\tan^{-1} x} + C.$

(ii)

Put $(x^2+1) = t$ so that $x^2 = (t-1)$ and $x\,dx = \frac{1}{2} dt$.

$$\therefore \int \frac{x^3}{(x^2+1)^3} dx = \int \frac{x^2 \cdot x}{(x^2+1)^3} dx$$

$$= \frac{1}{2}\int \frac{(t-1)}{t^3} dt = \frac{1}{2}\int \frac{1}{t^2} dt - \frac{1}{2}\int \frac{1}{t^3} dt$$

$$= \frac{-1}{2t} + \frac{1}{4t^2} + C = \frac{-1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C = \frac{-(1+2x^2)}{4(x^2+1)^2} + C.$$

OR

(ii)

Put $x^3 = t$ so that $3x^2 dx = dt$ or $x^2 dx = \frac{1}{3} dt$.

$$\therefore \int \frac{dx}{x \cdot \sqrt{x^6-1}} = \int \frac{x^2}{x^3 \cdot \sqrt{x^6-1}} dx$$

[multiplying numerator and denominator by x^2]

$$= \frac{1}{3}\int \frac{1}{t\sqrt{t^2-1}} dt = \frac{1}{3}\sec^{-1} t + C = \frac{1}{3}\sec^{-1} x^3 + C.$$