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PRACTICE PAPER 05 (2023-24)

CHAPTER 05 CONTINUITY AND DIFFERENTIABILITY

(ANSWERS)

SUBJECT: MATHEMATICS | **MAX. MARKS : 40**

CLASS : XII | **DURATION : $1\frac{1}{2}$ hrs**

General Instructions:

(i). **All** questions are compulsory.

(ii). This question paper contains 2**0** questions divided into five Sections A, B, C, D and E.

(iii). **Section A** comprises of **10 MCQs** of **1 mark** each. **Section B** comprises of 4 questions of **2 marks** each. **Section C** comprises of 3 questions of **3 marks** each. **Section D** comprises of 1 question of **5 marks** each and **Section E** comprises of 2 Case Study Based Questions of **4 marks** each.

(iv). There is no overall choice.

(v). Use of Calculators is not permitted

SECTION – A

Questions 1 to 10 carry 1 mark each.

1. A function f is said to be continuous for $x \in R$, if

(a) it is continuous at $x = 0$ (b) differentiable at $x = 0$

(c) continuous at two points (d) differentiable for $x \in R$

Ans: (d), as differentiable functions is continuous also.

2. A function $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, x \neq 0 \\ 2k \qquad , x = 0 \end{cases}$ is continuous at $x = 0$ for

(a) $k = 1$ (b) $k = 2$ (c) $k = \frac{1}{2}$ (d) $k = \frac{3}{2}$

Ans: (a), as $\lim\limits_{x \to 0}\left(\dfrac{\sin x}{x} + \cos x\right) = 1 + 1 = 2 = 2k \Rightarrow k = 1$

3. If $y = \tan^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{1}{1 + x^4}$ (b) $\dfrac{-2x}{1 + x^4}$ (c) $\dfrac{-1}{1 + x^4}$ (d) $\dfrac{x^2}{1 + x^4}$

Ans: (b) $\dfrac{-2x}{1 + x^4}$

(b), $y = \tan^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right) = \tan^{-1}\left(\dfrac{\pi}{4}\right) - \tan^{-1} x^2$

$$y' = 0 - \frac{1}{1 + x^4} \cdot 2x = \frac{-2x}{1 + x^4}$$

4. If $y = \sin^{-1}\left(\dfrac{3x}{2} - \dfrac{x^3}{2}\right)$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{3}{\sqrt{4 - x^2}}$ (b) $\dfrac{-3}{\sqrt{4 - x^2}}$ (c) $\dfrac{1}{\sqrt{4 - x^2}}$ (d) $\dfrac{-1}{\sqrt{4 - x^2}}$

Ans: (a) $\dfrac{3}{\sqrt{4 - x^2}}$

(a), as $y = \sin^{-1}\left\{3 \cdot \frac{x}{2} - 4 \cdot \left(\frac{x}{2}\right)^3\right\} = 3\sin^{-1}\frac{x}{2}$

$$\therefore \frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-\frac{x^2}{4}}} \cdot \frac{1}{2} = \frac{3}{\sqrt{4-x^2}}$$

5. If $y = Ae^{5x} + Be^{-5x}$ then $\frac{d^2y}{dx^2}$ is equal to

(a) $25y$ (b) $5y$ (c) $-25y$ (d) $10y$

Ans: (a), as $y' = 5Ae^{5x} - 5Be^{-5x}$
and $y'' = 25Ae^{5x} + 25Be^{-5x} = 25y$

6. Derivative of $\sin x$ with respect to $\log x$, is

(a) $\frac{x}{\cos x}$ (b) $\frac{\cos x}{x}$ (c) $x \cos x$ (d) $x^2 \cos x$

Ans: (c), let $y = \sin x$ and $t = \log x$,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \cos x \times \frac{x}{1} = x\cos x$$

7. The function 'f' defined by $f(x) = f(x) = \begin{cases} \frac{x^3-8}{x-2}, & x \neq 2 \\ 12, & x = 2 \end{cases}$ is

(a) not continuous at $x = 2$ (b) continuous at $x = 2$
(c) not continuous at $x = 3$ (d) not continuous at $x = -2$

Ans: (b) continuous at $x = 2$

8. If $x = at^2$, $y = 2at$, then $\frac{d^2y}{dx^2}$ is

(a) $\frac{1}{t}$ (b) $-\frac{1}{t^2}$ (c) at^2 (d) $-\frac{1}{2at^3}$

Ans: (d) $-\frac{1}{2at^3}$

In the following questions 9 and 10, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:
(a)Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).
(b)Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).
(c)Assertion (A) is true but reason (R) is false.
(d)Assertion (A) is false but reason (R) is true.

9. **Assertion (A):** $f(x) = |x - 3|$ is continuous at $x = 0$.
Reason (R): $f(x) = |x - 3|$ is differentiable at $x = 0$.
Ans: (b) Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).

10. **Assertion (A):** Every differentiable function is continuous but converse is not true.
Reason (R): Function $f(x) = |x|$ is continuous.
Ans: (c) Assertion (A) is true but reason (R) is false

SECTION – B

Questions 11 to 14 carry 2 marks each.

11. Find the value of *k* so that the function *f* defined by $f(x)=\begin{cases} kx+1, if & x\le\pi \\ \cos x, if & x>\pi \end{cases}$ is continuous at $x = \pi$.

Ans:

The given function is

$$f(x)=\begin{cases} kx+1, & \text{if } x\le\pi \\ \cos x, & \text{if } x>\pi \end{cases}$$

The given function *f* is continuous at $x = \pi$.

$$\Rightarrow \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x\to\pi^-}(kx+1) = \lim_{x\to\pi^+}\cos x = k\pi + 1$$

$$\lim_{h\to 0}[k(\pi-h)+1] = \lim_{h\to 0}\cos(\pi+h) = k\pi+1$$

$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1 \Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Hence, the required value of *k* is $-\frac{2}{\pi}$.

12. Find $\frac{dy}{dx}$, if $\sin y + x = \log x$

Ans: Consider $\sin y + x = \log x$

Differentiating both sides with respect to *x*, we get

$$\cos y\frac{dy}{dx}+1=\frac{1}{x} \Rightarrow \cos y.\frac{dy}{dx}=\frac{1}{x}-1 \Rightarrow \frac{dy}{dx}=\frac{1-x}{x\cos y}$$

13. Differentiate $5^{\sin x}$, with respect to *x*.

Ans: $\frac{d}{dx}(5^{\sin x}) = 5^{\sin x}\log_e 5.\frac{d}{dx}(\sin x) = 5^{\sin x}\log_e 5.\cos x$

14. Discuss the continuity of the function *f*(*x*) at $x = 1$, defined by $f(x)=\begin{cases} \frac{3}{2}-x, if & \frac{1}{2}\le x<1 \\ \frac{3}{2}, if & x=1 \\ \frac{3}{2}+x, if & 1<x\le 2 \end{cases}$

Ans:

$$\underset{x=1}{\text{LHL}} = \lim_{x\to 1^-}\left(\frac{3}{2}-x\right) = \frac{3}{2}-1 = \frac{1}{2}$$

$$\underset{x=1}{\text{RHL}} = \lim_{x\to 1^+}\left(\frac{3}{2}+x\right) = \frac{3}{2}+1 = \frac{5}{2}$$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Differentiate $x^{x\cos x}+\frac{x^2+1}{x^2-1}$ w.r.t. *x*

Ans:

Let $y = x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{x\cos x}\right) + \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) \quad ...(i)$$

Consider $u = x^{x\cos x} \Rightarrow \log u = x\cos x \,.\, \log x$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cos x \,.\, \frac{1}{x} + x\log x\,(-\sin x) + \log x \,.\, 1 \,.\, \cos x$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}\,[\cos x - x\log x \,.\, \sin x + \log x \cos x] \quad ...(ii)$$

Consider, $$\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{(x^2-1)\cdot\frac{d}{dx}(x^2+1) - (x^2+1)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)\,2x - (x^2+1)2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \quad ...(iii)$$

Substituting from (ii) and (iii) in (i), we get

$$\frac{dy}{dx} = x^{x\cos x}\,[\cos x - x\log x \,.\, \sin x + \log x \,.\, \cos x\,] - \frac{4x}{(x^2-1)^2}.$$

16. Show that the function $f(x) = |x - 3|$, $x \in R$ is continuous but not differentiable at $x = 3$.

Ans:

Given function $f(x) = |x-3| = \begin{cases} x-3, & x \geq 3 \\ -x+3, & x < 3 \end{cases}$

For continuity at $x = 3$,

$$\underset{x=3}{\text{LHL}} = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}\{-(3-h)+3\} = \lim_{h\to 0} h = 0$$

$$\underset{x=3}{\text{RHL}} = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}\{(3+h)-3\} = \lim_{h\to 0} h = 0$$

$$f(3) = 3 - 3 = 0$$

As $\underset{x=3}{\text{LHL}} = \underset{x=3}{\text{RHL}} = f(3)$,

Hence, function is continuous at $x = 3$.

For differentiability at $x = 3$,

$$\underset{x=3}{\text{LHD}} = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h} = \lim_{h\to 0}\frac{(-3+h+3)-(0)}{-h} = \lim_{h\to 0}\frac{h}{-h} = \lim_{h\to 0}(-1) = -1$$

$$\underset{x=3}{\text{RHD}} = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{(3+h-3)-(0)}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0}(1) = 1$$

As $\underset{x=3}{\text{LHD}} \neq \underset{x=3}{\text{RHD}}$. Hence, function is not derivable (differentiable) at $x = 3$.

17. Find $\dfrac{dy}{dx}$, if $y = \tan^{-1}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

Ans:

Let $x = \cos\theta$, then $$y = \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$\Rightarrow y = \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x \Rightarrow \frac{dy}{dx} = 0 - \frac{-1}{2\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}}.$$

SECTION – D

Questions 18 carry 5 marks.

18. Let $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & if \quad x<\dfrac{\pi}{2}\\ a, & if \quad x=\dfrac{\pi}{2}\\ \dfrac{b(1-\sin x)}{(\pi-2x)^2}, & if \quad x>\dfrac{\pi}{2}\end{cases}$. If $f(x)$ be a continuous function at $x=\frac{\pi}{2}$, find a and b.

Ans:

If function is continuous at $x=\frac{\pi}{2}$, then $\underset{x=\frac{\pi}{2}}{\text{LHL}} = \underset{x=\frac{\pi}{2}}{\text{RHL}} = f\left(\frac{\pi}{2}\right)$ ...(i)

$$\underset{x=\frac{\pi}{2}}{\text{LHL}} = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right) = \lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0}\frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)} = \lim_{h\to 0}\frac{1+\cos^2 h+\cos h}{3(1+\cos h)} = \frac{1+1+1}{3(1+1)} = \frac{1}{2} \quad ...(ii)$$

$$\underset{x=\frac{\pi}{2}}{\text{RHL}} = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right) = \lim_{h\to 0}\frac{b\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2} = \lim_{h\to 0}\frac{b(1-\cos h)}{(\pi-\pi-2h)^2}$$

$$= \lim_{h\to 0}\frac{b(1-\cos h)}{4h^2} = \lim_{h\to 0}\frac{b.2\sin^2\frac{h}{2}}{4h^2} = \lim_{h\to 0}\frac{b}{8}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 = \frac{b}{8}\times 1 = \frac{b}{8} \quad ...(iii)$$

Substituting from (ii) and (iii) in (i), we get

$$\frac{1}{2} = \frac{b}{8} = a \Rightarrow a = \frac{1}{2}, b = 4$$

Hence, for $a=\frac{1}{2}$ and $b=4$ function is continuous at $x=\frac{\pi}{2}$.

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. Case-Study 3:

Sumit has a doubt in the continuity and differentiability problem, but due to COVID-19 he is unable to meet with his teachers or friends. So he decided to ask his doubt with his friends Sunita and Vikram with the help of video call. Sunita said that the given function is continuous for all the real value of x while Vikram said that the function is continuous for all the real value of x except at x = 3.

The given function is f (x) = $\dfrac{x^2-9}{x-3}$

Based on the above information, answer the following questions:

(a) Whose answer is correct? (1)

(b) Find the derivative of the given function with respect to x. (1)

(c) Find the value of f ′(3). (1)

(d) Find the second differentiation of the given function with respect to x. (1)

Ans: (a) Vikram (b) 1 (c) 1 (d) 0

20. A potter made a mud vessel, where the shape of the pot is based on f(x) = |x – 3| + | x – 2|, where f(x) represents the height of the pot.

(a) When x > 4 What will be the height in terms of x? (1)

(b) What is $\frac{dy}{dx}$ at x = 3? (1)

(c) When the x value lies between (2, 3) then the function is _____ (1)

(d) If the potter is trying to make a pot using the function f(x) = [x], will he get a pot or not? Why? (1)

Ans: (a) 2x – 5

(b) function is not differentiable

(c) 1

(d) No, because it is not continuous