10 Short Questions

1. **What is the definition of continuity of a function at a point?**
   A function $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

2. **What is the left-hand limit of $f(x) = |x|$ at $x = 0$?**
   $\lim_{x\to 0^-} |x| = 0$.

3. **Check the continuity of $f(x) = x^2$ at $x = 0$.**
   Solution:
   - $f(0) = 0$
   - $\lim_{x\to 0} f(x) = 0$.
   - Hence, $f(x)$ is continuous at $x = 0$.

4. **What is the relationship between continuity and differentiability?**
   If a function is differentiable at a point, it is continuous at that point.

5. **Determine the points of discontinuity of $f(x) = [x]$, the greatest integer function.**
   $f(x)$ is discontinuous at all integers.

6. **State whether $f(x) = |x|$ is continuous at $x = 0$.**
   Yes, it is continuous as the left-hand limit, right-hand limit, and function value are equal.

7. **What is the condition for continuity at an endpoint of an interval?**
   The one-sided limit should equal the function value at the endpoint.

8. **Is $f(x) = 1/x$ continuous at $x = 0$? Why?**
   No, because the function is not defined at $x = 0$.

9. **For $f(x) = x^3$, check if it is continuous at $x = 2$.**
   $\lim_{x\to 2} f(x) = f(2) = 8$. Hence, $f(x)$ is continuous at $x = 2$.

10. **What is the derivative of $f(x) = x^n$?**
    $f'(x) = n \cdot x^{n-1}$.

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10 Long Questions

1. **Prove that $f(x) = 5x - 3$ is continuous at $x = 0$, $x = -3$, and $x = 5$.**
   Solution:
   - At $x = 0$: $f(0) = -3$, $\lim_{x\to 0} f(x) = -3$.
   - At $x = -3$: $f(-3) = -18$, $\lim_{x\to -3} f(x) = -18$.
   - At $x = 5$: $f(5) = 22$, $\lim_{x\to 5} f(x) = 22$.
   - In all cases, $\lim_{x\to c} f(x) = f(c)$.

2. **Check the continuity of $f(x) = |x|$ at $x = 0$.**
   Solution:
   - $f(0) = 0$.
   - Left-hand limit: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$.
   - Right-hand limit: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$.
   - Since all limits and the function value are equal, $f(x)$ is continuous at $x = 0$.

3. **Prove that $f(x) = x^3 + x^2 - 1$ is continuous for all $x \in \mathrm{R}$.**
   Solution:
   - $f(x)$ is a polynomial function.
   - Polynomials are continuous everywhere.
   - Hence, $f(x)$ is continuous for all real numbers.
4. **Find the points of discontinuity of** $f(x) = \begin{cases} x+2, & \text{if } x \leq 1 \\ x-2, & \text{if } x > 1 \end{cases}$

   Solution:
   - For $x < 1$ and $x > 1$, $f(x)$ is continuous.
   - At $x = 1$, left-hand limit: $\lim_{x \to 1^-} f(x) = 3$.
   - Right-hand limit: $\lim_{x \to 1^+} f(x) = -1$.
   - Left-hand limit $\neq$ right-hand limit, so $f(x)$ is discontinuous at $x = 1$.
5. **Differentiate $f(x) = \sin(\cos x)$.**
   Solution:
   - Let $y = \sin(\cos x)$.
   - By chain rule, $\frac{dy}{dx} = \cos(\cos x) \cdot (-\sin x)$.
   - Hence, $\frac{dy}{dx} = -\sin x \cdot \cos(\cos x)$.
6. **Verify the continuity of $f(x) = \tan x$ for $x \neq \frac{\pi}{2} + n\pi$.**
   Solution:
   - $\tan x$ is defined and continuous for $x \neq \frac{\pi}{2} + n\pi$.
   - As the left-hand and right-hand limits exist and equal the function value, $f(x)$ is continuous.
7. **Prove that $f(x) = |x| - |x+1|$ is discontinuous at $x = -1$.**
   Solution:
   - At $x = -1$, left-hand limit: -1, right-hand limit: -2.
   - Left-hand limit $\neq$ right-hand limit, so $f(x)$ is discontinuous.
8. **Differentiate $f(x) = x^x$.**
   Solution:
   - Let $y = x^x$. Taking log: $\ln y = x \ln x$.
   - Differentiate: $\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + 1$.
   - $\frac{dy}{dx} = x^x(\ln x + 1)$.
9. **Find the derivative of $f(x) = e^{\sin x}$.**
   Solution:
   - Let $y = e^{\sin x}$.
   - $\frac{dy}{dx} = e^{\sin x} \cdot \cos x$.
10. **Check the continuity of** $f(x) = \begin{cases} x^2, & \text{if } x \leq 2 \\ 3x+1, & \text{if } x > 2 \end{cases}$ **at $x = 2$.**

    Solution:
    - Left-hand limit: $\lim_{x \to 2^-} f(x) = 4$.
    - Right-hand limit: $\lim_{x \to 2^+} f(x) = 7$.
    - Since the limits are not equal, $f(x)$ is not continuous at $x = 2$.

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1. Prove that $f(x) = \cos(x^2)$ is continuous for all $x \in \mathbb{R}$.

Solution:

1. $f(x) = \cos(x^2)$ is a composite function. Let $g(x) = x^2$ and $h(x) = \cos(x)$.
2. Both $g(x)$ and $h(x)$ are continuous for all $x \in \mathbb{R}$.
3. By the theorem of continuity of composite functions, $f(x) = h(g(x))$ is continuous for all x.
4. Therefore, $f(x) = \cos(x^2)$ is continuous for all $x \in \mathbb{R}$.

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2. Show that the greatest integer function $f(x) = [x]$ is discontinuous at all integers.

Solution:

1. The greatest integer function is defined as $f(x) = [x]$, which gives the largest integer less than or equal to x.
2. At an integer c, the left-hand limit is:

   $$\lim_{x \to c^-} f(x) = c - 1.$$

   The right-hand limit is:

   $$\lim_{x \to c^+} f(x) = c.$$

3. Since the left-hand limit $\neq$ right-hand limit, $f(x)$ is discontinuous at all integers.

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3. Discuss the continuity of $f(x) = \frac{1}{x}$ for $x \neq 0$.

Solution:

1. The function $f(x) = \frac{1}{x}$ is defined for all $x \neq 0$.
2. For $x > 0$, $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. Similarly, for $x < 0$, the limit also equals $\frac{1}{c}$.
3. Since the left-hand and right-hand limits equal the function value for all $x \neq 0$, $f(x)$ is continuous for $x \neq 0$.

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4. Check the differentiability of $f(x) = |x|$ at $x = 0$.

Solution:

1. $f(x) = |x|$ can be written as:

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

2. The left-hand derivative at $x = 0$ is:

$$f'(0^-) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = -1.$$

3. The right-hand derivative at $x = 0$ is:

$$f'(0^+) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1.$$

4. Since the left-hand derivative $\neq$ right-hand derivative, $f(x)$ is not differentiable at $x = 0$.

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5. Prove that $f(x) = e^x$ is continuous and differentiable for all $x \in \mathbb{R}$.

Solution:

1. $f(x) = e^x$ is an exponential function defined for all $x \in \mathbb{R}$.
2. Continuity: Let $c \in \mathbb{R}$. The limit is:

$$\lim_{x \to c} e^x = e^c = f(c).$$

   Hence, $f(x)$ is continuous for all x.

3. Differentiability: The derivative of $f(x) = e^x$ is $f'(x) = e^x$, which exists for all x.
4. Therefore, $f(x) = e^x$ is continuous and differentiable for all $x \in \mathbb{R}$.

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6. Find the derivative of $f(x) = \tan(x^2)$.

Solution:

1. Let $y = \tan(x^2)$.
2. By the chain rule:

$$\frac{dy}{dx} = \sec^2(x^2) \cdot \frac{d}{dx}(x^2).$$

3. $\frac{dy}{dx} = \sec^2(x^2) \cdot 2x = 2x \sec^2(x^2)$.

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7. Determine the continuity of (f(x) = \begin{cases}

x^2, & \text{if } x \leq 1, \ 2x - 1, & \text{if } x > 1. \end{cases})**

Solution:

1. For $x \leq 1$, $f(x) = x^2$ is continuous. For $x > 1$, $f(x) = 2x - 1$ is continuous.
2. At $x = 1$:
   - Left-hand limit: $\lim_{x \to 1^-} f(x) = 1^2 = 1$.

- Right-hand limit: $\lim_{x \to 1^+} f(x) = 2(1) - 1 = 1$.
- $f(1) = 1$.

3. Since all limits are equal, $f(x)$ is continuous at $x = 1$.

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8. Differentiate $f(x) = \log(\sin x)$.

Solution:

1. Let $y = \log(\sin x)$.
2. By the chain rule:
$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x.$$
3. Simplify:
$$\frac{dy}{dx} = \cot x.$$

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9. Find the derivative of $f(x) = x^x$.

Solution:

1. Let $y = x^x$. Taking log:
$$\ln y = x \ln x.$$
2. Differentiate both sides:
$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1.$$
3. Multiply through by $y = x^x$:
$$\frac{dy}{dx} = x^x(\ln x + 1).$$

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10. Prove that $f(x) = \sin x + \cos x$ is continuous and differentiable for all x.

Solution:

1. $f(x) = \sin x + \cos x$ is a sum of trigonometric functions, which are continuous for all x.
2. Continuity: At any $c \in \mathrm{R}$, $\lim_{x \to c} f(x) = \sin c + \cos c = f(c)$. Hence, $f(x)$ is continuous.
3. Differentiability: The derivative is:
$$f'(x) = \cos x - \sin x,$$
which exists for all x.
4. Therefore, $f(x)$ is continuous and differentiable for all x.