5. Oscillations

Can you recall?

1. What do you mean by linear motion and angular motion?
2. Can you give some practical examples of oscillations in our daily life?
3. What do you know about restoring force?
4. All musical instruments make use of oscillations, can you identify, where?
5. Why does a ball floating on water bobs up and down, if pushed down and released?

5.1 Introduction:

Oscillation is a very common and interesting phenomenon in the world of Physics. In our daily life we come across various examples of oscillatory motion, like rocking of a cradle, swinging of a swing, motion of the pendulum of a clock, the vibrations of a guitar or violin string, up and down motion of the needle of a sewing machine, the motion of the prongs of a vibrating tuning fork, oscillations of a spring, etc. In these cases, the motion repeated after a certain interval of time is a periodic motion. Here the motion of an object is mostly to and fro or up and down.

Oscillatory motion is a periodic motion. In this chapter, we shall see that the displacement, velocity and acceleration for this motion can be represented by sine and cosine functions. These functions are known as harmonic functions. Therefore, an oscillatory motion obeying such functions is called harmonic motion. After studying this chapter, you will be able to understand the use of appropriate terminology to describe oscillations, simple harmonic motion (S.H.M.), graphical representations of S.H.M., energy changes during S.H.M., damping of oscillations, resonance, etc.

5.2 Explanation of Periodic Motion:

Any motion which repeats itself after a definite interval of time is called periodic motion. A body performing periodic motion goes on repeating the same set of movements. The time taken for one such set of movements is called its period or periodic time. At the end of each set of movements, the state of the body is the same as that at the beginning. Some examples of periodic motion are the motion of the moon around the earth and the motion of other planets around the sun, the motion of electrons around the nucleus, etc. As seen in Chapter 1, the uniform circular motion of any object is thus a periodic motion.

Another type of periodic motion in which a particle repeatedly moves to and fro along the same path is the *oscillatory* or *vibratory motion*. Every oscillatory motion is periodic but every periodic motion need not be oscillatory. Circular motion is periodic but it is not oscillatory.

The simplest form of oscillatory periodic motion is the simple harmonic motion in which every particle of the oscillating body moves to and fro, about its mean position, along a certain fixed path. If the path is a straight line, the motion is called *linear simple harmonic motion* and if the path is an arc of a circle, it is called *angular simple harmonic motion*. The smallest interval of time after which the to and fro motion is repeated is called its period (T) and the number of oscillations completed per unit time is called the frequency (n) of the periodic motion.

Can you tell?

Is the motion of a leaf of a tree blowing in the wind periodic?

5.3 Linear Simple Harmonic Motion (S.H.M.):

Place a rectangular block on a smooth frictionless horizontal surface. Attach one end of a spring to a rigid wall and the other end to the block as shown in Fig. 5.1. Pull the block of mass m towards the right and release it. The block will begin its to and fro motion on either side of its equilibrium position. This motion is linear simple harmonic motion.

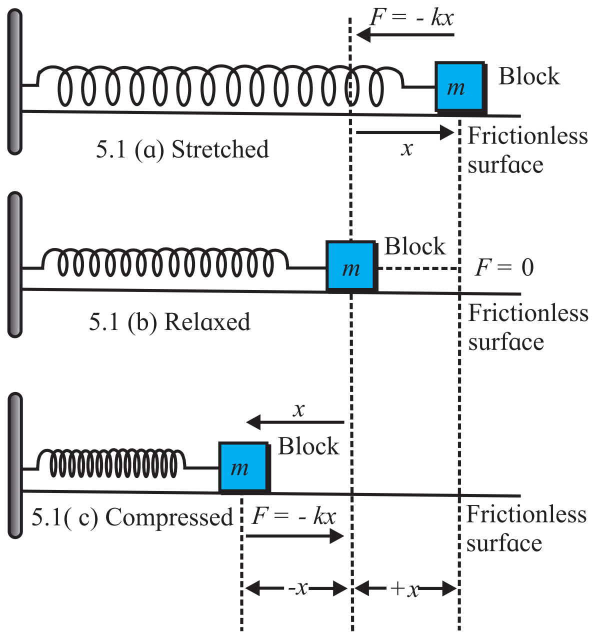


Fig. 5.1 (a), (b) and (c): Spring mass oscillator.

Remember this

For such a motion, as a convention, we shall always measure the displacement from the mean position. Also, as the entire motion is along a single straight line, we need not use vector notation (only $\pm$ signs will be enough).

Fig. 5.1(b) shows the equilibrium position in which the spring exerts no force on the block. If the block is displaced towards the right from its equilibrium position, the force exerted by the spring on the block is directed towards the left [Fig. 5.1(a)]. On account of its elastic properties, the spring tends to regain its original shape and size and therefore it exerts a restoring force on the block. This is responsible to bring it back to the original position. This force is proportional to the displacement but its direction is opposite to that of the displacement. If x is the displacement, the restoring force f is given by,

$$f = -kx \qquad \text{--- (5.1)}$$

where, k is a constant that depends upon the elastic properties of the spring. It is called the *force constant*. The negative sign indicates that the force and displacement are oppositely directed.

If the block is displaced towards left from its equilibrium position, the force exerted by the spring on the block is directed towards the right and its magnitude is proportional to the displacement from the mean position. (Fig. 5.1(c))

Thus, $f = -kx$ can be used as the equation of motion of the block.

Now if the block is released from the rightmost position, the restoring force exerted by the spring accelerates it towards its equilibrium position. The acceleration (a) of the block is given by,

$$a = \frac{f}{m} = -\left(\frac{k}{m}\right)x \qquad \text{--- (5.2)}$$

where, m is mass of the block. This shows that the acceleration is also proportional to the displacement and its direction is opposite to that of the displacement, i.e., the force and acceleration are both directed towards the mean or equilibrium position.

As the block moves towards the mean position, its speed starts increasing due to its acceleration, but its displacement from the mean position goes on decreasing. When the block returns to its mean position, the displacement and hence force and acceleration are zero. The speed of the block at the mean position becomes maximum and hence its kinetic energy attains its maximum value. Thus, the block does not stop at the mean position, but continues to move beyond the mean position towards the left. During this process, the spring is compressed and it exerts a restoring force on the block towards right. Once again, the force and displacement are oppositely directed. This opposing force retards the motion of the block, so that the speed goes on reducing and finally it becomes zero. This position is shown in Fig. 5.1(c). In this position the displacement from the mean position and restoring force are maximum. This force now accelerates the block towards the right, towards the equilibrium position. The process goes on repeating that causes the block to oscillate on either side of its equilibrium (mean) position. Such oscillatory motion along a straight path is called linear simple harmonic motion (S.H.M.). *Linear S.H.M. is defined as the linear periodic motion of a body, in which force (or acceleration) is always directed towards the mean position and its magnitude is proportional to the displacement from the mean position.*

Use your brain power

If there is friction between a block and the resting surface, how will it govern the motion of the block?

Remember this

A complete oscillation is when the object goes from one extreme to other and back to the initial position.

The conditions required for simple harmonic motion are:

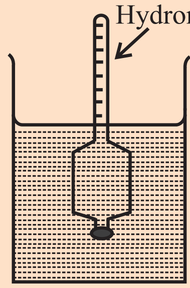
1. Oscillation of the particle is about a fixed point.
2. The net force or acceleration is always directed towards the fixed point.
3. The particle comes back to the fixed point due to restoring force.

Harmonic oscillation is that oscillation which can be expressed in terms of a single harmonic function, such as $x = a \sin\omega t$ or $x = a \cos\omega t$

Non-harmonic oscillation is that oscillation which cannot be expressed in terms of single harmonic function. It may be a combination of two or more harmonic oscillations such as $x = a \sin \omega t + b \sin 2\omega t$, etc.

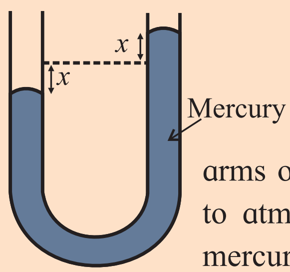
Activity

Some experiments described below can be performed in the classroom to demonstrate S.H.M. Try to write their equations.



(a) A hydrometer is immersed in a glass jar filled with water. In the equilibrium position it floats vertically in water. If it is slightly depressed and released, it bobs up and down performing linear S.H.M.

(b) A U-tube is filled with a sufficiently long column of mercury. Initially when both the arms of U tube are exposed to atmosphere, the level of mercury in both the arms is the same. Now, if the level of mercury in one of the arms is depressed slightly and released, the level of mercury in each arm starts moving up and down about the equilibrium position, performing linear S.H.M.



5.4 Differential Equation of S.H.M. :

In a linear S.H.M., the force is directed towards the mean position and its magnitude is directly proportional to the displacement of the body from mean position. As seen in Eq. (5.1),

$$f = -kx$$

where k is force constant and x is displacement from the mean position.

According to Newton's second law of motion,

$$f = ma \quad \therefore \; ma = -kx \qquad \text{--- (5.3)}$$

The velocity of the particle is, $v = \frac{dx}{dt}$

and its acceleration, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Substituting it in Eq. (5.3), we get

$$m\frac{d^2x}{dt^2} = -kx$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{--- (5.4)}$$

Substituting $\frac{k}{m} = \omega^2$, where ω is the angular frequency,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{--- (5.5)}$$

Eq. (5.5) is the differential equation of linear S.H.M.

Can you tell?

Why is the symbol ω and also the term angular frequency used for a linear motion?

Example 5.1 A body of mass 0.2 kg performs linear S.H.M. It experiences a restoring force of 0.2 N when its displacement from the mean position is 4 cm. Determine (i) force constant (ii) period of S.H.M. and (iii) acceleration of the body when its displacement from the mean position is 1 cm.

Solution: (i) Force constant,

$$k = f/x$$

$$= (0.2)/\ 0.04 = 5 \text{ N/m}$$

(ii) Period $T = 2\pi/\omega$

$$= 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2}{5}} = 0.4\pi \text{ s}$$

(iii) Acceleration

$$a = -\omega^2 x = -\frac{k}{m}x = -\frac{5}{0.2} \times 0.04 = -1 \text{ m s}^{-2}$$

5.5 Acceleration (a), Velocity (v) and Displacement (x) of S.H.M.:

We can obtain expressions for the acceleration, velocity and displacement of a particle performing S.H.M. by solving the differential equation of S.H.M. in terms of displacement x and time t.

From Eq. (5.5), we have $\frac{d^2x}{dt^2} + \omega^2 x = 0$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2 x \qquad \text{--- (5.6)}$$

But $a = \frac{d^2x}{dt^2}$ is the acceleration of the particle performing S.H.M.

$$\therefore \quad a = -\omega^2 x \qquad \text{--- (5.7)}$$

This is the expression for acceleration in terms of displacement x.

From Eq. (5.6), we have $\frac{d^2x}{dt^2} = -\omega^2 x$

$$\therefore \quad \frac{d}{dt}\left(\frac{dx}{dt}\right) = -\omega^2 x$$

$$\therefore \frac{dv}{dt} = -\omega^2 x$$

$$\therefore \frac{dv}{dx}\frac{dx}{dt} = -\omega^2 x$$

$$\therefore v\frac{dv}{dx} = -\omega^2 x$$

$$\therefore v\,dv = -\omega^2 x\,dx$$

Integrating both the sides, we get

$$\int v\,dv = -\omega^2 \int x\,dx$$

$$\therefore \frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + C, \qquad \text{--- (5.8)}$$

where C is the constant of integration.

Let A be the maximum displacement (amplitude) of the particle in S.H.M.

When the particle is at the extreme position, velocity (v) is zero.

Thus, at $x = \pm A$, $v = 0$

Substituting in Eq. (5.8), we get

$$0 = -\frac{\omega^2 A^2}{2} + C$$

$$\therefore C = +\frac{\omega^2 A^2}{2} \qquad \text{--- (5.9)}$$

Using C in Eq. (5.8), we get

$$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 A^2}{2}$$

$$\therefore v^2 = \omega^2\left(A^2 - x^2\right)$$

$$\therefore v = \pm\omega\sqrt{A^2 - x^2} \qquad \text{--- (5.10)}$$

This is the expression for the velocity of a particle performing linear S.H.M. in terms of displacement x.

Substituting $v = \frac{dx}{dt}$ in Eq. (5.10), we get

$$\frac{dx}{dt}=\omega\sqrt{A^2-x^2}$$

$$\therefore \frac{dx}{\sqrt{A^2-x^2}}=\omega dt$$

Integrating both the sides, we get

$$\int\frac{dx}{\sqrt{A^2-x^2}}=\omega\int dt$$

$$\sin^{-1}\left(\frac{x}{A}\right)=\omega t+\phi \quad \text{--- (5.11)}$$

Here ϕ is the constant of integration. To know ϕ, we need to know the value of x at any instance of time t, most convenient being $t = 0$.

$$\therefore x = A\sin(\omega t+\phi) \quad \text{--- (5.12)}$$

This is the general expression for the displacement (x) of a particle performing linear S.H.M. at time t. Let us find expressions for displacement for two particular cases.

Case (i) If the particle starts S.H.M. from the mean position, $x = 0$ at $t = 0$

Using Eq. (5.11), we get $\phi=\sin^{-1}\left(\frac{x}{A}\right)=0$ or π

Substituting in Eq. (5.12), we get

$$x=\pm A\sin(\omega t) \quad \text{--- (5.13)}$$

This is the expression for displacement at any instant if the particle starts S.H.M. from the mean position. Positive sign to be chosen if it starts towards positive and negative sign for starting towards negative.

Case (ii) If the particle starts S.H.M. from the extreme position, $x=\pm A$ at $t=0$

$$\therefore \phi=\sin^{-1}\left(\frac{x}{A}\right)=\frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

Substituting in Eq. (5.12), we get

$$x=A\sin\left(\omega t+\frac{\pi}{2}\right) \text{ or } x=A\sin\left(\omega t+\frac{3\pi}{2}\right)$$

$$\therefore \ x=\pm A\cos(\omega t) \quad \text{--- (5.14)}$$

This is the expression for displacement at any instant, if the particle starts S.H.M. from the extreme position. Positive sign for starting from positive extreme position and negative sign for starting from the negative extreme position.

In the cases (i) and (ii) above, we have used the phrase, "if the particle starts S.H.M….." More specifically, it is not the particle that starts its S.H.M., but we (the observer) start counting the time t from that instant. The particle is already performing its motion. We start recording the time as per our convenience. In other words, $t = 0$ (or initial condition) is always subjective to the observer.

Expressions of displacement (x), velocity (v) and acceleration (a) at time t:

From Eq. (5.12), $x = A\sin(\omega t+\phi)$

$$\therefore v=\frac{dx}{dt}=A\omega\cos(\omega t+\phi)$$

$$\therefore a=\frac{dv}{dt}=-A\omega^2\sin(\omega t+\phi)$$

Example 5.2: A particle performs linear S.H.M. of period 4 seconds and amplitude 4 cm. Find the time taken by it to travel a distance of 1 cm from the positive extreme position.

Solution: $x = A\sin(\omega t+\phi)$

Since particle performs S.H.M. from positive extreme position, $\phi = \frac{\pi}{2}$ and from data

$$x = A-1=3\,\text{cm}$$

$$\therefore 3=4\sin\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right)$$

$$\therefore \ \frac{3}{4}=\cos\frac{2\pi}{4}t=\cos\frac{\pi}{2}t$$

$$\therefore \ \frac{\pi}{2}t=41.4^0=\left(41.4\times\frac{\pi}{180}\right)^c \ \therefore \ t=0.46 \text{ s}$$

$$\left[\text{Or, } \frac{180}{2}t=41.4 \ \therefore \ t=0.46 \text{ s}\right]$$

Example 5.3: A particle performing linear S.H.M. with period 6 second is at the positive extreme position at $t = 0$. The particle is found to be at a distance of 3 cm from this position at time $t = 7$s, before reaching the mean position. Find the amplitude of S.H.M.

Solution: $x = A\sin(\omega t + \phi)$

Since particle starts (t = 0) from positive extreme position, ϕ = π/2 and $x = A - 3$

$$\therefore x = A\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\therefore A - 3 = A\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right)$$

$$\therefore \frac{A-3}{A} = \sin\left(\frac{2\pi}{6}\times 7 + \frac{\pi}{2}\right)$$

$$\therefore \frac{A-3}{A} = \sin\left(\frac{7\pi}{3} + \frac{\pi}{2}\right)$$

$$\therefore \frac{A-3}{A} = \sin\left(\frac{\pi}{3} + \frac{\pi}{2}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\therefore 2A - 6 = A$$

$$\therefore A = 6\,\text{cm}$$

Example 5.4: The speeds of a particle performing linear S.H.M. are 8 cm/s and 6 cm/s at respective displacements of 6 cm and 8 cm. Find its period and amplitude.

Solution:

$$\text{v} = \omega\sqrt{\left(A^2 - x^2\right)}$$

$$\therefore \frac{8}{6} = \frac{\omega\sqrt{\left(A^2 - 6^2\right)}}{\omega\sqrt{\left(A^2 - 8^2\right)}} \text{ or } \frac{4}{3} = \frac{\omega\sqrt{\left(A^2 - 36\right)}}{\omega\sqrt{\left(A^2 - 64\right)}}$$

$$\therefore \; A = 10\,\text{cm}$$

$$\text{v}_1 = \omega\sqrt{\left(A^2 - x_1^{\,2}\right)}$$

$$\therefore 8 = \frac{2\pi}{T}\sqrt{\left(10^2 - 6^2\right)} \therefore \; 8 = \frac{2\pi}{T}8$$

$$\therefore \; T = 6.284 \text{ s}$$

Extreme values of displacement (x), velocity (v) and acceleration (a):

1) Displacement: The general expression for displacement x in S.H.M. is $x = A\,\sin\left(\omega t + \phi\right)$

At the mean position, $\left(\omega t + \phi\right) = 0$ or π

$\therefore \; x_{min} = 0.$

Thus, at the mean position, the displacement of the particle performing S.H.M. is minimum (i.e. zero).

At the extreme position, $\left(\omega t + \phi\right) = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$$\therefore x = A\,\sin\left(\omega t + \phi\right)$$

$$\therefore x = \pm A\,\sin\frac{\pi}{2} \quad \therefore \; x_{max} = \pm A$$

Thus, at the extreme position the displacement of the particle performing S.H.M. is maximum.

2) Velocity: According to Eq. (5.10) the magnitude of velocity of the particle performing S.H.M. is $\text{v} = \pm\omega\sqrt{A^2 - x^2}$

At the mean position, $x = 0 \therefore \text{v}_{max} = \pm A\omega$.

Thus, the velocity of the particle in S.H.M. is maximum at the mean position.

At the extreme position, $x = \pm A \;\therefore\; \text{v}_{min} = 0$.

Thus, the velocity of the particle in S.H.M. is minimum at the extreme positions.

3) Acceleration: The magnitude of the acceleration of the particle in S.H.M is $\omega^2 x$

At the mean position $x = 0$, so that the acceleration is minimum. $\therefore\; a_{min} = 0$.

At the extreme positions $x = \pm A$, so that the acceleration is maximum $a_{max} = \mp\omega^2 A$

Can you tell?

1. State at which point during an oscillation the oscillator has zero velocity but positive acceleration?
2. During which part of the simple harmonic motion velocity is positive but the displacement is negative, and vice versa?
3. During which part of the oscillation the two are along the same direction?

Example 5.5: The maximum velocity of a particle performing S.H.M. is 6.28 cm/s. If the length of its path is 8 cm, calculate its period.

Solution:

$$\text{v}_{max} = 6.28\frac{\text{cm}}{\text{s}} = 2\pi\frac{\text{cm}}{\text{s}} \text{ and } A = 4\,\text{cm}$$

$$\text{v}_{max} = A\omega = A\frac{2\pi}{T}$$

$$\therefore\; 2\pi = 4\frac{2\pi}{T}$$

$$\therefore\; T = 4\,\text{s}$$

Example 5.6: The maximum speed of a particle performing linear S.H.M is 0.08 m/s. If its maximum acceleration is 0.32 m/s², calculate its (i) period and (ii) amplitude.

Solution:

(i) $\frac{a_{max}}{v_{max}} = \frac{A\omega^2}{A\omega} = \omega = \frac{2\pi}{T} \therefore \frac{0.32}{0.08} = \frac{2\pi}{T}$

$\therefore \quad T = 1.57\, s$

(ii) $v_{max} = A\omega = A\frac{2\pi}{T} \quad \therefore A = 2\,\text{cm}$

5.6: Amplitude(A), Period(T) and Frequency (n) of S.H.M. :

5.6.1 Amplitude of S.H.M.:

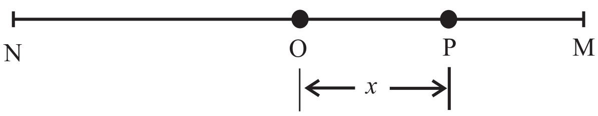


Fig. 5.2 S.H.M. of a particle.

Consider a particle P performing S.H.M. along the straight line MN (Fig. 5.2). The centre O of MN is the mean position of the particle.

The displacement of the particle as given by Eq. (5.12) is $x = A\sin(\omega t + \phi)$

The particle will have its maximum displacement when $\sin(\omega t + \phi) = \pm 1$, i.e., when $x = \pm A$. This distance A is called the amplitude of S.H.M.

The maximum displacement of a particle performing S.H.M. from its mean position is called the amplitude of S.H.M.

5.6.2 Period of S.H.M.:

The time taken by the particle performing S.H.M. to complete one oscillation is called the period of S.H.M.

Displacement of the particle at time t is given by $x = A\sin(\omega t + \phi)$

After a time $t' = \left(t + \frac{2\pi}{\omega}\right)$ the displacement will be

$$x' = A\sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$$\therefore x' = A\sin(\omega t + 2\pi + \phi)$$

$$\therefore x' = A\sin(\omega t + \phi)$$

This result shows that the particle is at the same position after a time $\frac{2\pi}{\omega}$. That means, the particle completes one oscillation in time $\frac{2\pi}{\omega}$. It can be shown that $t = T = \frac{2\pi}{\omega}$ is the minimum time after which it repeats.

Hence its period T is given by $T = \frac{2\pi}{\omega}$

From Eq.(5.4) and Eq.(5.5)

$$\omega^2 = \frac{k}{m} = \frac{\text{force per unit displacement}}{\text{mass}}$$

$$= \text{acceleration per unit displacement}$$

$$\therefore T = \frac{2\pi}{\sqrt{\text{acceleration per unit displacement}}}$$

Also, $T = 2\pi\sqrt{\frac{m}{k}}$ --- (5.15)

5.6.3 Frequency of S.H.M.:

The number of oscillations performed by a particle performing S.H.M. per unit time is called the frequency of S.H.M.

In time T, the particle performs one oscillation. Hence in unit time it performs $\frac{1}{T}$ oscillations.

Hence, frequency n of S.H.M. is given by

$n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ --- (5.16)

Combination of springs: A number of springs of different spring constants can be combined in series (Figure A) or in parallel (Figure B) or both.

Series combination (Figure A): In this case, all the springs are connected one after the other forming a single chain. Consider an arrangement of two such springs of spring constants k_1 and k_2. If the springs are massless, each will have the same stretching force as f. For vertical arrangement, it will be the weight mg. If e_1 and e_2 are the respective extensions, we can write,

$$f = k_1 e_1 = k_2 e_2 \therefore e_1 = \frac{f}{k_1} \text{ and } e_2 = \frac{f}{k_2}$$

The total extension is

$$e = e_1 + e_2 = f\left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

If k_s is the effective spring constant (as if there is a single spring that gives the same total extension for the same force), we can write,

$$e = \frac{f}{k_s} = f\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \therefore \frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$$

For a number of such (massless) springs, in series, $\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots\ldots = \sum_i \left(\frac{1}{k_i}\right)$

For only two massless springs of spring constant k each, in series, $k_s = \frac{k_1 k_2}{k_1 + k_2} = \frac{\text{Product}}{\text{Sum}}$

For n such identical massless springs, in series, $k_s = \frac{k}{n}$

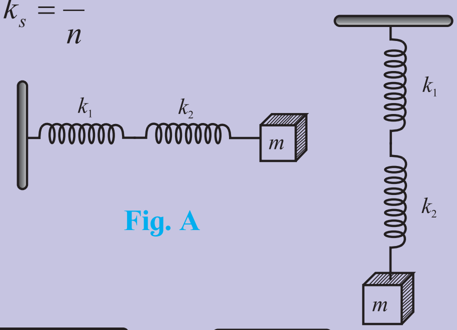


Fig. A

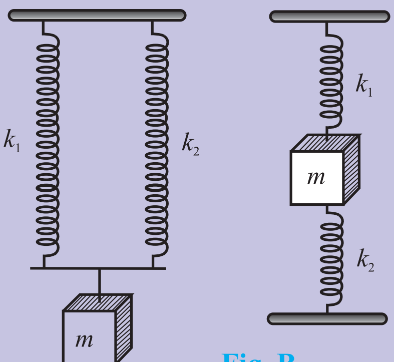


Fig. B

Parallel combination (Figure B): In such a combination, all the springs are connected between same two points, one of them is the support and at the other end, the stretching force f is applied at a *suitable* point. Irrespective of their spring constants, each spring will now have the same extension e. The springs now share the force such that in the equilibrium position, the total restoring force is equal and opposite to the stretching force f.

Let $f_1 = k_1 e, f_2 = k_2 e, \ldots\ldots$ be the individual restoring forces.

If k_p is the effective spring constant, a single spring of this spring constant will be stretched by the same extension e, by the same stretching force f.

$\therefore f = k_p e = f_1 + f_2 + \ldots = k_1 e + k_2 e + \ldots$

$\therefore k_p = k_1 + k_2 + \ldots = \sum k_i$

For m such identical massless springs of spring constant k each, in parallel, $k_p = mk$

5.7 Reference Circle Method:

Figure 5.3 shows a rod rotating along a vertical circle in the x-y plane. If the rod is illuminated parallel to x-axis from either side by a linear source parallel to the rod, as shown in the Fig. 5.3, the shadow (projection) of the rod will be produced on the y-axis. The tip of this shadow can be seen to be oscillating about the origin, along the y-axis.

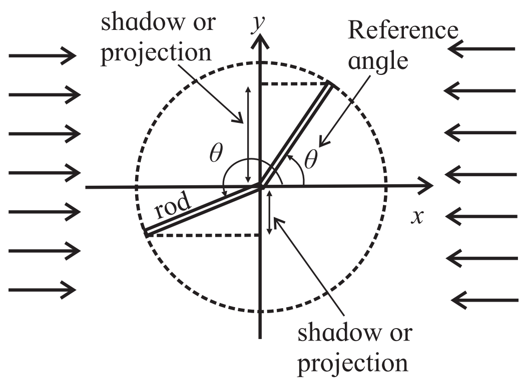


Fig 5.3: Projection of a rotating rod.

We shall now prove that motion of the tip of the projection is an S.H.M. if the corresponding motion of the tip of the rod is a U.C.M. For this, we should take the projections of displacement, velocity, etc. on *any* reference diameter and confirm that we get the corresponding quantities for a linear S.H.M.

Figure 5.4 shows the anticlockwise uniform circular motion of a particle P, with centre at the origin O. Its angular positions are decided with the reference OX. It means, if the particle is at E, the angular position is zero, at

F it is 90° $=\frac{\pi}{2}^c$, at G it is 180° = π^c, and so on. If it comes to E again, it will be 360° = $2\pi^c$ (and not zero). Let $\vec{r} = OP$ be the position vector of this particle.

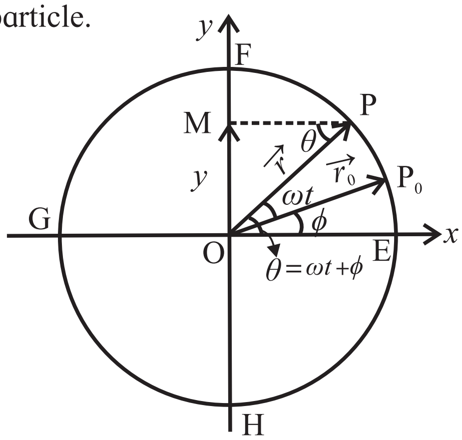


Fig 5.4: S.H.M. as projection of a U.C.M.

At t = 0, let the particle be at P_0 with reference angle ϕ. During time t, it has angular displacement ωt. Thus, the reference angle at time t is $\theta = (\omega t + \phi)$. Let us choose the diameter FH along y-axis as the reference diameter and label OM as the projection of $\vec{r}$ = OP on this.

***Projection of displacement*:** At time t, we get the projection or the position vector OM = OP $\sin\theta = y = r\sin(\omega t + \phi)$. This is the equation of linear S.H.M. of amplitude r. The term ω can thus be understood as the angular velocity of the reference circular motion. For linear S.H.M. we may call it the *angular frequency* as it decides the periodicity of the S.H.M. In the next section, you will come to know that the phase angle $\theta = (\omega t + \phi)$ of the circular motion can be used to be the phase of the corresponding S.H.M.

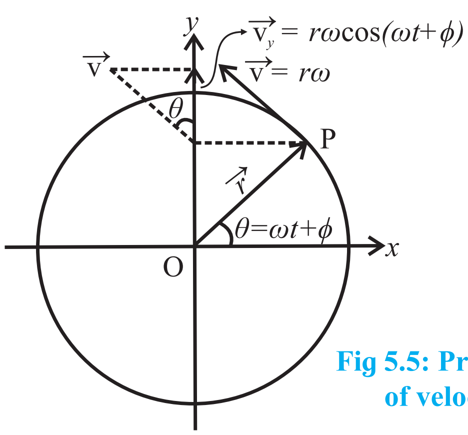


Fig 5.5: Projection of velocity.

***Projection of velocity*:** Instantaneous velocity of the particle P in the circular motion is the tangential velocity of magnitude $r\omega$ as shown in the Fig. 5.5.

Its projection on the reference diameter will be $v_y = r\omega\cos\theta = r\omega\cos(\omega t + \phi)$. This is the expression for the velocity of a particle performing a linear S.H.M.

***Projection of acceleration*:** Instantaneous acceleration of the particle P in circular motion is the radial or centripetal acceleration of magnitude $r\omega^2$, directed towards O. Its projection on the reference diameter will be $a_y = -r\omega^2\sin\theta = -r\omega^2\sin(\omega t + \phi) = -\omega^2 y$.

Again, this is the corresponding acceleration for the linear S.H.M.

From this analogy it is clear that projection of any quantity for a uniform circular motion gives us the corresponding quantity of linear S.H.M. This analogy can be verified for any diameter as the reference diameter. Thus, the projection of a U.C.M. on any diameter is an S.H.M.

5.8 Phase in S.H.M.:

Phase in S.H.M. (or for any motion) is basically the state of oscillation. In order to know the state of oscillation in S.H.M., we need to know the displacement (position), the direction of velocity and the oscillation number (during which oscillation) at that instant of time. Knowing only the displacement is not enough, because at a given position there are two possible directions of velocity (except the extreme positions), and it repeats for successive oscillations. Knowing only velocity is not enough because there are two different positions for the same velocity (except the mean position). Even after this, both these repeat for the successive oscillations.

Hence, to know the phase, we need a quantity that is continuously changing with time. It is clear that all the quantities of linear S.H.M. (x, v, a etc) are the projections taken on a diameter, of the respective quantities for the reference circular motion. The angular displacement $\theta = (\omega t + \phi)$ can thus be used as the phase of S.H.M. as it varies continuously with time. In this case, it will be called as the *phase angle*.

Special cases:

(i) Phase $\theta = 0$ indicates that the particle is at the mean position, moving to the positive, during the beginning of the first oscillation. Phase angle $\theta = 360^0$ or $2\pi^c$ is the beginning of the second oscillation, and so on for the successive oscillations.

(ii) Phase $\theta = 180^0$ or π^c indicates that during its first oscillation, the particle is at the mean position and moving to the negative. Similar state in the second oscillation will have phase $\theta = (360+180)^0$ or $(2\pi+\pi)^c$, and so on for the successive oscillations.

(iii) Phase $\theta = 90^0$ or $\left(\frac{\pi}{2}\right)^c$ indicates that the particle is at the positive extreme position during first oscillation. For the second oscillation it will be $\theta = (360+90)^0$ or $\left(2\pi + \frac{\pi}{2}\right)^c$, and so on for the successive oscillations.

(iv) Phase $\theta = 270^0$ or $\left(\frac{3\pi}{2}\right)^c$ indicates that the particle is at the negative extreme position during the first oscillation. For the second oscillation it will be $\theta = (360+270)^0$ or $\left(2\pi + \frac{3\pi}{2}\right)^c$, and so on for the successive oscillations.

Example 5.7: Describe the state of oscillation if the phase angle is 1110^0.

Solution: $1110^0 = 3\times 360^0 + 30^0$

3×360^0 plus something indicates 4^{th} oscillation. Now, $A\sin 30^0 = \frac{A}{2}$

Thus, phase angle 1110^0 indicates that during its 4^{th} oscillation, the particle is at $+A/2$ and moving to the positive extreme.

Example 5.8: While completing its third oscillation during linear S.H.M., a particle is at $\frac{-\sqrt{3}}{2}A$, heading to the mean position. Determine the phase angle.

Solution:

$A\sin\theta_1 = \frac{-\sqrt{3}}{2}A \therefore \theta_1 = \left(\pi + \frac{\pi}{3}\right)^c$ or $\left(2\pi - \frac{\pi}{3}\right)^c$

From negative side, the particle is heading to the mean position. Thus, the phase angle is in the fourth quadrant for that oscillation.

$$\therefore \theta_1 = \left(2\pi - \frac{\pi}{3}\right)^c$$

As it is the third oscillation, phase

$$\theta = 2\times 2\pi + \theta_1 \therefore \theta = 4\pi + \left(2\pi - \frac{\pi}{3}\right)$$

$$= 6\pi - \frac{\pi}{3} = \left(\frac{17\pi}{3}\right)^c$$

5.9. Graphical Representation of S.H.M.:

(a) Particle executing S.H.M., starting from mean position, towards positive:

As the particle starts from the mean position Fig (5.6), towards positive, $\phi = 0$

$\therefore$ displacement $x = A\sin\omega t$

Velocity $v = A\omega\cos\omega t$

Acceleration $a = -A\omega^2 \sin\omega t$

(t)	0	$T/4$	$T/2$	$3T/4$	T	$5T/4$
(θ^*)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
(x)	0	A	0	$-A$	0	A
(v)	$A\omega$	0	$-A\omega$	0	$A\omega$	0
(a)	0	$-A\omega^2$	0	$A\omega^2$	0	$-A\omega^2$

* phase $\theta = \omega t + \phi$

Conclusions from the graphs:

- Displacement, velocity and acceleration of S.H.M. are periodic functions of time.
- Displacement time curve and acceleration time curves are sine curves and velocity time curve is a cosine curve.
- There is phase difference of $\pi/2$ radian between displacement and velocity.
- There is phase difference of $\pi/2$ radian between velocity and acceleration.

- There is phase difference of π radian between displacement and acceleration.
- Shapes of all the curves get repeated after 2π radian or after a time T.

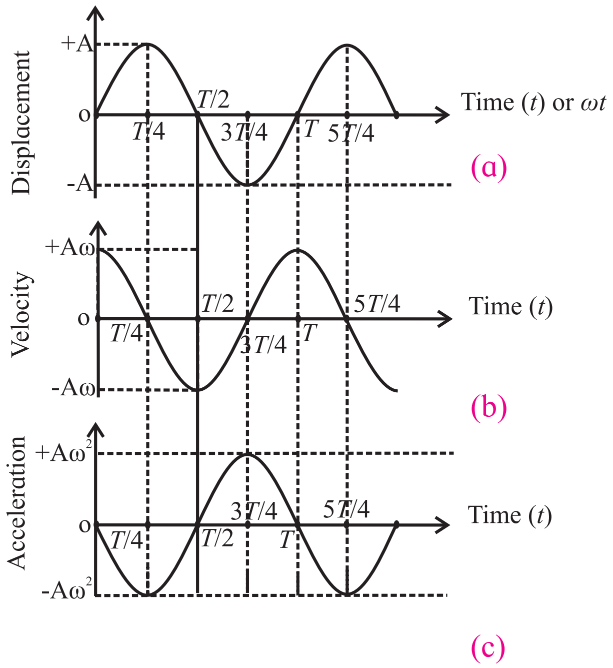


Fig. 5.6: (a) Variation of displacement with time, (b) Variation of velocity with time, (c) Variation of acceleration with time.

((b) Particle performing S.H.M., starting from the positive extreme position.

As the particle starts from the positive extreme position Fig. (5.7), $\phi = \frac{\pi}{2}$

$\therefore$ displacement, $x = A\sin\left(\omega t + \pi/2\right) = A\cos\omega t$

Velocity, $v = \frac{dx}{dt} = \frac{d\left(A\cos\omega t\right)}{dt} = -A\omega\sin\left(\omega t\right)$

Acceleration,

$$a = \frac{dv}{dt} = \frac{d\left(-A\omega\sin\left(\omega t\right)\right)}{dt} = -A\omega^2\cos\left(\omega t\right)$$

(t)	0	$T/4$	$T/2$	$3T/4$	T	$5T/4$
(θ)*	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3 π
(x)	A	0	$-A$	0	A	0
(v)	0	$-A\omega$	0	$A\omega$	0	$-A\omega$
(a)	$-A\omega^2$	0	$A\omega^2$	0	$-A\omega^2$	0

*(Phase $\theta = \omega t + \phi$)

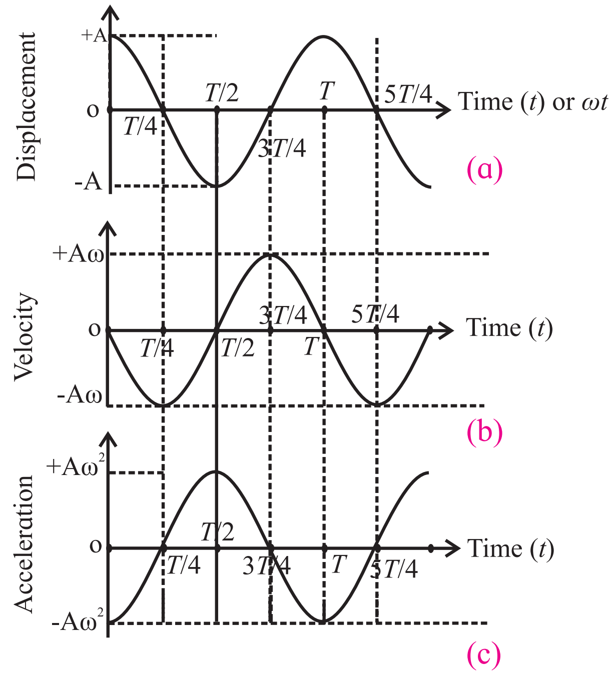


Fig. 5.7: (a) Variation of displacement with time, (b) Variation of velocity with time, (c) Variation of acceleration with time.

5.10 Composition of two S.H.M.s having same period and along the same path:

Consider a particle subjected simultaneously to two S.H.M.s having the same period and along same path (let it be along the x-axis), but of different amplitudes and initial phases. The resultant displacement at any instant is equal to the vector sum of its displacements due to both the S.H.M.s at that instant.

Equations of displacement of the two S.H.M.s along same straight line (x-axis) are

$x_1 = A_1 \sin(\omega t + \phi_1)$ and $x_2 = A_2 \sin(\omega t + \phi_2)$

The resultant displacement (x) at any instant (t) is given by $x = x_1 + x_2$

$x = A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2)$

$\therefore x = A_1 \sin\omega t \cos\phi_1 + A_1 \cos\omega t \sin\phi_1$
$\quad + A_2 \sin\omega t \cos\phi_2 + A_2 \cos\omega t \sin\phi_2$

A_1, A_2, ϕ_1 and ϕ_2 are constants and ωt is variable. Thus, collecting the constants together,

$x = (A_1 \cos\phi_1 + A_2 \cos\phi_2)\sin\omega t +$
$\quad (A_1 \sin\phi_1 + A_2 \sin\phi_2)\cos\omega t$

As A_1, A_2, ϕ_1 and ϕ_2 are constants, we can combine them in terms of another convenient constants R and δ as

$$R\cos\delta = A_1\cos\phi_1 + A_2\cos\phi_2 \quad \text{--- (5.17)}$$

and $\quad R\sin\delta = A_1\sin\phi_1 + A_2\sin\phi_2 \quad$ --- (5.18)

$\therefore x = R(\sin\omega t\cos\delta + \cos\omega t\sin\delta)$

$\therefore \; x = R\sin(\omega t + \delta)$

This is the equation of an S.H.M. of the same angular frequency (hence, the same period) but of amplitude R and initial phase δ. It shows that the combination (superposition) of two linear S.H.M.s of the same period and occurring a long the same path is also an S.H.M.

Resultant amplitude,

$$R = \sqrt{(R\sin\delta)^2 + (R\cos\delta)^2}$$

Substituting from Eq. (5.17) and Eq. (5.18), we get

$R^2 = A_1^{\,2} + A_2^{\,2} + 2A_1A_2\cos(\phi_1 - \phi_2)$

$\therefore R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$ - (5.19)

Initial phase (δ) of the resultant motion: Dividing Eq. (5.18) by Eq. (5.17), we get

$$\frac{R\sin\delta}{R\cos\delta} = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

$$\therefore \tan\delta = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

$$\therefore \; \delta = \tan^{-1}\left(\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}\right) \quad \text{--- (5.20)}$$

Special cases: (i) If the two S.H.M.s are in phase, $(\phi_1 - \phi_2) = 0^\circ$, $\therefore \cos(\phi_1 - \phi_2) = 1$.

$\therefore R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2} = \pm(A_1 + A_2)$. Further, if $A_1 = A_2 = A$, we get $R = 2A$

(ii) If the two S.H.M.s are 90° out of phase, $(\phi_1 - \phi_2) = 90^\circ \; \therefore \cos(\phi_1 - \phi_2) = 0$.

$\therefore \; R = \sqrt{A_1^2 + A_2^2}$ Further, if $A_1 = A_2 = A$, we get, $R = \sqrt{2}\,A$

(iii) If the two S.H.M.s are 180° out of phase, $(\phi_1 - \phi_2) = 180^\circ \; \therefore \cos(\phi_1 - \phi_2) = -1$

$\therefore R = \sqrt{A_1^2 + A_2^2 - 2A_1A_2} \; \therefore R = |A_1 - A_2|$

Further, if $A_1 = A_2 = A$, we get $R = 0$

Activity

Tie a string horizontally tight between two vertical supports. To this string, tie three pendula, two of them (A and B) of equal lengths. Third one (C) need not have the same length, but not very different. Oscillate the pendula A and B in a plane perpendicular to the horizontal string. It will be observed that pendulum C also starts oscillating in the same plane, with the same period as those of A and B.

With this system and procedure, we are imposing two S.H.M.s of the same period. The resultant energy transfers through the strings into the third pendulum C and it starts oscillating. Special cases (i), (ii) and (iii) above can be verified by making suitable changes.

5.11: Energy of a Particle Performing S.H.M.:

While performing an S.H.M., the particle possesses speed (hence kinetic energy) at all the positions except at the extreme positions. In spite of the presence of a restoring force (except at the mean position), the particle occupies various positions. This is an indication that work is done and the system has potential energy (elastic - in the case of a spring, gravitational - for a pendulum, magnetic - for a magnet, etc.). Total energy of the particle performing an S.H.M. is thus the sum of its kinetic and potential energies.

Consider a particle of mass m, performing a linear S.H.M. along the path MN about the mean position O. At a given instant, let the particle be at P, at a distance x from O.

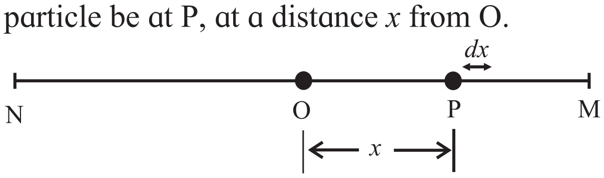


Fig. 5.8: Energy in an S.H.M.

Velocity of the particle in S.H.M. is given as $\text{v} = \omega\sqrt{A^2 - x^2} = A\omega\cos(\omega t + \phi)$, where x is the displacement of the particle performing S.H.M. and A is the amplitude of S.H.M.

Thus, the kinetic energy,

$$E_k = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) = \frac{1}{2}k\left(A^2 - x^2\right) \quad \text{--- (5.21)}$$

This is the kinetic energy at displacement x.

At time t, it is

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2\left(\omega t + \phi\right)$$

$$= \frac{1}{2}kA^2\cos^2\left(\omega t + \phi\right) \quad \text{--- (5.22)}$$

Thus, with time, it varies as $\cos^2\theta$.

The restoring force acting on the particle at point P is given by $f = -kx$ where k is the force constant. Suppose that the particle is displaced further by an infinitesimal displacement dx against the restoring force f. The external work done (dW) during this displacement is

$$dW = f\left(-dx\right) = -kx\left(-dx\right) = kx\,dx$$

The total work done on the particle to displace it from O to P is given by

$$W = \int_0^x dW = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

This should be the potential energy (P.E.) E_p of the particle at displacement x.

$$\therefore E_P = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \quad \text{--- (5.23)}$$

At time t, it is

$$E_P = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2\left(\omega t + \phi\right)$$

$$= \frac{1}{2}mA^2\omega^2\sin^2\left(\omega t + \phi\right) \quad \text{--- (5.23a)}$$

Thus, with time, it varies as $\sin^2\theta$.

The total energy of the particle is the sum of its kinetic energy and potential energy.

$$\therefore E = E_k + E_p$$

Using Eq. (5.21) and Eq. (5.23), we get

$$E = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) + \frac{1}{2}m\omega^2x^2$$

$$E = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2 = \frac{1}{2}m\left(v_{max}\right)^2 \quad \text{---(5.24)}$$

This expression gives the total energy of the particle at point P. As m, ω and A are constant, the total energy of the particle at any point P is constant (independent of x and t). In other words, the energy is conserved in S.H.M.

If n is the frequency of S.H.M., $\omega = 2\pi n$. Using this in Eq. (5.24), we get

$$E = \frac{1}{2}m\left(2\pi n\right)^2A^2 = 2\pi^2n^2A^2m$$

$$= 2\pi^2m\frac{A^2}{T^2} \quad \text{--- (5.25)}$$

Thus, the total energy in S.H.M. is directly proportional to (a) the mass of the particle (b) the square of the amplitude (c) the square of the frequency (d) the force constant, and inversely proportional to square of the period.

Can you tell?

To start a pendulum swinging, usually you pull it slightly to one side and release.

- What kind of energy is transferred to the mass in doing this?
- Describe the energy changes that occur when the mass is released.
- Is/are there any other way/ways to start the oscillations of a pendulum? Which energy is supplied in this case/cases?

Special cases: (i) At the mean position, $x = 0$ and velocity is maximum.

Hence $E = \left(E_k\right)_{max} = \frac{1}{2}m\omega^2A^2$ and potential energy $\left(E_p\right)_{min} = 0$

(ii) At the extreme positions, the velocity of the particle is zero and $x = \pm A$

Hence $E = \left(E_p\right)_{max} = \frac{1}{2}m\omega^2A^2$ and kinetic energy $\left(E_k\right)_{min} = 0$

As the particle oscillates, the energy changes between kinetic and potential. At the mean position, the energy is entirely kinetic; while at the extreme positions, it is entirely potential. At other positions the energy is partly kinetic and partly potential. However, the total energy is always conserved.

(iii) If $K.E. = P.E.$,

$$\frac{1}{2}m\omega^2\left(A^2 - x^2\right) = \frac{1}{2}m\omega^2x^2 \quad \therefore x = \frac{\pm A}{\sqrt{2}}$$

Thus at $x = \frac{\pm A}{\sqrt{2}}$, the K.E. = P.E. = $\frac{E}{2}$ for a particle performing linear S.H.M.

(iv) At $x = \frac{\pm A}{2}$, P.E. $= \frac{1}{2}kx^2 = \frac{1}{4}\left(\frac{1}{2}kA^2\right) = \frac{E}{4}$

$\therefore$ K.E. $= 3(\text{P.E})$

Thus, at $x = \frac{\pm A}{2}$, the energy is 25% potential and 75% kinetic.

The variation of K.E. and P.E. with displacement in S.H.M. is shown in Fig. (5.9)

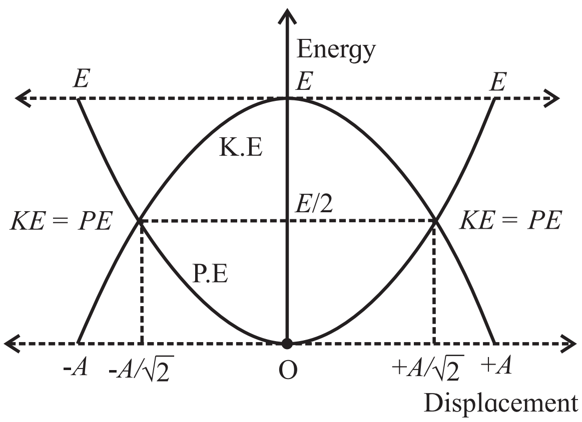


Fig. 5.9: Energy in S.H.M.

Example 5.9: The total energy of a particle of mass 200 g, performing S.H.M. is 10^{-2} J. Find its maximum velocity and period if the amplitude is 7 cm.

Solution:

$$E = \frac{1}{2}m\omega^2 A^2 \therefore \quad E = \frac{1}{2}m\left(v_{max}\right)^2$$

$$\therefore v_{max} = \sqrt{\frac{2E}{m}}$$

$$\therefore v_{max} = \sqrt{\frac{2\times 10^{-2}}{0.2}} = 0.3162\,\text{m/s}$$

$$v_{max} = \omega A = \frac{2\pi}{T}A \therefore T = \frac{2\pi A}{v_{max}} = 1.391\,\text{s}$$

5.12 Simple Pendulum:

An ideal simple pendulum is a heavy particle suspended by a massless, inextensible, flexible string from a rigid support.

A practical simple pendulum is a small heavy (dense) sphere (called bob) suspended by a light and inextensible string from a rigid support.

The distance between the point of suspension and centre of gravity of the bob (point of oscillation) is called the length of the pendulum. Let m be the mass of the bob and T' be the tension in the string. The pendulum remains in equilibrium in the position OA, with the centre of gravity of the bob, vertically below the point of suspension O. If now the pendulum is displaced through a small angle θ (called angular amplitude) and released, it begins to oscillate on either side of the mean (equilibrium) position in a single vertical plane. We shall now show that the bob performs S.H.M. about the mean position for small angular amplitude θ.

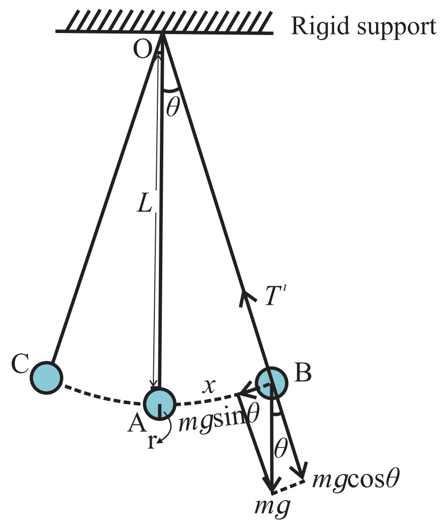


Fig.5.10: Simple pendulum.

In the displaced position (extreme position), two forces are acting on the bob.

(i) Force T' due to tension in the string, directed along the string, towards the support and

(ii) Weight mg, in the vertically downward direction.

At the extreme positions, there should not be any net force along the string. The component of mg can only balance the force due to tension. Thus, weight mg is resolved into two components;

(i) The component $mg\cos\theta$ along the string, which is balanced by the tension T' and

(ii) The component $mg\sin\theta$ perpendicular to the string is the restoring force acting on mass m tending to return it to the equilibrium position.

$\therefore$ Restoring force, $F = -\,mg\sin\theta$ --- (5.26)

As θ is very small ($\theta < 10°$), we can write

$\sin\theta \cong \theta^c \therefore F \cong -mg\theta$

From the Fig. 5.10, the small angle $\theta, = \frac{x}{L}$

$$\therefore\ F = -mg\frac{x}{L} \quad \text{--- (5.27)}$$

As m, g and L are constant, $F \propto -\,x$

Thus, for small displacement, the restoring force is directly proportional to the displacement and is oppositely directed.

Hence the bob of a simple pendulum performs linear S.H.M. for small amplitudes. From Eq. (5.15), the period T of oscillation of a pendulum from can be given as,

$$= \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{acceleration\ per\ unit\ displacement}}$$

Using Eq. (5.27), $F = -mg\frac{x}{L}$

$$\therefore\ ma = -mg\frac{x}{L}$$

$$\therefore\ a = -g\frac{x}{L}\ \therefore\ \frac{a}{x} = -\frac{g}{L} = \frac{g}{L}\ (\text{in magnitude})$$

Substituting in the expression for T, we get,

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{--- (5.28)}$$

The Eq. (5.28) gives the expression for the time period of a simple pendulum. However, while deriving the expression the following assumptions are made.

(i) The amplitude of oscillations is very small (at least 20 times smaller than the length).

(ii) The length of the string is large and

(iii) During the oscillations, the bob moves along a single vertical plane.

Frequency of oscillation n of the simple pendulum is

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \text{--- (5.29)}$$

From the Eq. (5.28), we can conclude the following for a simple pendulum.

(a) The period of a simple pendulum is directly proportional to the square root of its length.

(b) The period of a simple pendulum is inversely proportional to the square root of acceleration due to gravity.

(c) The period of a simple pendulum does not depend on its mass.

(d) The period of a simple pendulum does not depend on its amplitude (for small amplitude).

These conclusions are also called the 'laws of simple pendulum'.

5.12.1 Second's Pendulum:

A simple pendulum whose period is two seconds is called *second's pendulum.*

$$\text{Period } T = 2\pi\sqrt{\frac{L}{g}}$$

$$\therefore \text{For a second's pendulum, } 2 = 2\pi\sqrt{\frac{L_s}{g}}$$

where L_s is the length of second's pendulum, having period $T = 2$s.

$$\therefore L_s = \frac{g}{\pi^2} \quad \text{--- (5.30)}$$

Using this relation, we can find the length of a second's pendulum at a place, if we know the acceleration due to gravity at that place. Experimentally, if L_s is known, it can be used to determine acceleration due to gravity g at that place.

Example 5.10: The period of oscillations of a simple pendulum increases by 10%, when its length is increased by 21 cm. Find its initial length and initial period.

Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

$$\therefore\ \frac{100}{110} = \sqrt{\frac{l_1}{l_2}}$$

$$\therefore\ \frac{10}{11} = \sqrt{\frac{l_1}{l_1 + 0.21}}$$

$$\therefore\ 1.21 l_1 = l_1 + 0.21 \therefore l_1 = 1\,\text{m}$$

$$\therefore\ \text{Period } T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{1}{9.8}}$$

$$= 2.007\ \text{s}\ (\pi = 3.142)$$

Activity

When you perform the experiment to determine the period of simple pendulum, it is recommended to keep the amplitude very small. But how small should it be? And why?

To find this it would be better to measure the time period for different angular amplitudes.

Let $T_0 = 2\pi\sqrt{\frac{L}{g}}$ be the period for (ideally) very small angular amplitude and T_θ be the period at higher angular amplitude θ. Experimentally determined values of the ratio $\frac{T_\theta}{T_0}$ are as shown in the table below.

θ	20°	45°	50°	70°	90°
$\frac{T_\theta}{T_0}$	1.02	1.04	1.05	1.10	1.18

It shows that the error in the time period is about 2% at amplitude of 20°, 5% at amplitude of 50°, 10% at amplitude of 70° and 18% at amplitude of 90°. Thus, the recommended maximum angular amplitude is less than 20°. It also helps us in restricting the oscillations in a single vertical plane.

Example 5.11: In summer season, a pendulum clock is regulated as a second's pendulum and it keeps correct time. During winter, the length of the pendulum decreases by 1%. How much will the clock gain or lose in one day. ($g = 9.8\ m/s^2$)

Solution: In summer, with period $T_s = 2$ s, the clock keeps correct time. Thus, in a day of 86400 seconds, the clock's pendulum should perform $\frac{86400}{2} = 43200$ oscillations, to keep correct time.

L_w = 1% less than summer $= 0.99L_s$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$\therefore T \propto \sqrt{L} \therefore \frac{T_w}{T_s} = \sqrt{\frac{L_w}{L_s}} \therefore \frac{T_w}{2} = \sqrt{0.99}$$

$$\therefore T_w = 1.99 \text{ s}$$

With this period, the pendulum will now perform $\frac{86400}{1.99} = 43417$ oscillations per day. Thus, it will gain 43417 - 43200 = 217 oscillations, per day.

Per oscillations the clock refers to 2 second. Thus, the time gained, per day = 217 × 2 = 434 second = 7 minutes, 14 second.

	Conical pendulum	Simple pendulum
1	Trajectory and the plane of the motion of the bob is a horizontal circle	Trajectory and the plane of motion of the bob is part of a vertical circle.
2	K.E. and gravitational P.E. are constant.	K.E. and gravitational P.E. are interconverted and their sum is conserved.
3	Horizontal component of the force due to tension is the necessary centripetal force (governing force).	Tangential component of the weight is the governing force for the energy conversions during the motion.
4	Period, $T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$	Period, $T = 2\pi\sqrt{\frac{L}{g}}$
5	String always makes a fixed angle with the horizontal and can never be horizontal.	With large amplitude, the string can be horizontal at some instances.
6	*During the discussion for both, we have ignored the stretching of the string and the energy spent for it. However, the string is always stretched otherwise it will never have tension (except at the extreme positions of the simple pendulum). Also, non-conservative forces like air resistance are neglected.*	

5.13: Angular S.H.M. and its Differential Equation:

Figure 5.11 shows a metallic disc attached centrally to a thin wire (preferably nylon or metallic wire) hanging from a rigid support. If the disc is slightly twisted about the axis along the wire, and released, it performs rotational motion partly in clockwise and anticlockwise (or opposite) sense. Such oscillations are called angular oscillations or torsional oscillations.

This motion is governed by the restoring torque in the wire, which is always opposite to the angular displacement. If its magnitude happens to be proportional to the corresponding angular displacement, we can call the motion to be *angular* S.H.M.

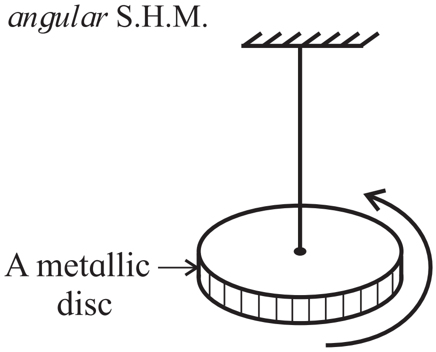


Fig. 5.11: Torsional (angular) oscillations.

Thus, for the angular S.H.M. of a body, the restoring torque acting upon it, for angular displacement θ, is

$$\tau \propto -\theta \text{ or } \tau = -c\theta \quad \text{--- (5.31)}$$

The constant of proportionality c is the restoring torque per unit angular displacement. If I is the moment of inertia of the body, the torque acting on the body is given by, $\tau = I\alpha$ Where α is the angular acceleration. Using this in Eq. (5.31) we get, $I\alpha = -c\theta$

$$\therefore I\frac{d^2\theta}{dt^2} + c\theta = 0 \quad \text{--- (5.32)}$$

This is the differential equation for angular S.H.M. From this equation, the angular acceleration α can be written as,

$$\alpha = \frac{d^2\theta}{dt^2} = -\frac{c\theta}{I}$$

Since c and I are constants, the angular acceleration α is directly proportional to θ and its direction is opposite to that of the angular displacement. Hence, this oscillatory motion is called angular S.H.M.

Angular S.H.M. is defined as the oscillatory motion of a body in which the torque for angular acceleration is directly proportional to the angular displacement and its direction is opposite to that of angular displacement.

The time period T of angular S.H.M. is given by, $T = \frac{2\pi}{\omega}$

$$= \frac{2\pi}{\sqrt{\frac{\text{angular acceleration per unit}}{\text{angular displacement}}}}$$

5.13.1 Magnet Vibrating in Uniform Magnetic Field:

If a bar magnet is freely suspended in the plane of a uniform magnetic field, it remains in equilibrium with its axis parallel to the direction of the field. If it is given a small angular displacement θ (about an axis passing through its centre, perpendicular to itself and to the field) and released, it performs angular oscillations Fig. (5.12).

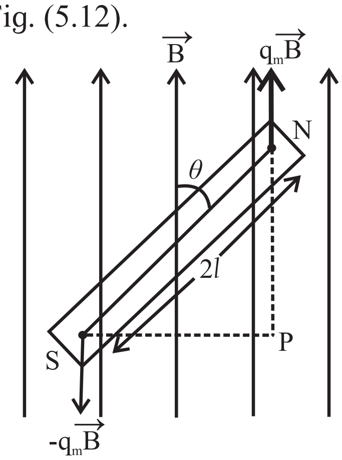


Fig. 5.12: Magnet vibrating in a uniform magnetic field.

Let μ be the magnetic dipole moment and B the magnetic field. In the deflected position, a restoring torque acts on the magnet, that tends to bring it back to its equilibrium position. [*Here we used the symbol μ for the magnetic dipole moment as the symbol m is used for mass*].

The magnitude of this torque is $\tau = \mu B\sin\theta$

If θ is small, $\sin\theta \cong \theta^c \quad \therefore \tau = \mu B\theta$

For clockwise angular displacement θ, the restoring torque is in the anticlockwise direction.

$$\therefore \quad \tau = I\alpha = -\mu B\theta$$

where I is the moment of inertia of the bar magnet and α is its angular acceleration.

$$\therefore \alpha = -\left(\frac{\mu B}{I}\right)\theta \qquad \text{--- (5.33)}$$

Since μ, B and I are constants, Eq. (5.33) shows that angular acceleration is directly proportional to the angular displacement and directed opposite to the angular displacement. Hence the magnet performs angular S.H.M.

The period of vibrations of the magnet is given by

$$T = \frac{2\pi}{\sqrt{\frac{\text{angular acceleration per unit}}{\text{angular displacement}}}}$$

$$= \frac{2\pi}{\sqrt{\alpha/\theta}}$$

$$\therefore T = 2\pi\sqrt{\frac{I}{\mu B}} \qquad \text{--- (5.34)}$$

Example 5.12: A bar magnet of mass 120 g, in the form of a rectangular parallelepiped, has dimensions l = 40 mm, b = 10 mm and h = 80 mm. With the dimension h vertical, the magnet performs angular oscillations in the plane of a magnetic field with period π s. If its magnetic moment is 3.4 A m^2, determine the influencing magnetic field.

Solution: $T = 2\pi\sqrt{\frac{I}{\mu B}}$ $\quad \therefore \pi = 2\pi\sqrt{\frac{I}{\mu B}}$

$$\therefore B = \frac{4I}{\mu}$$

For a bar magnet, moment of inertia

$$I = M\left(\frac{l^2 + b^2}{12}\right)$$

$$\therefore I = 0.12\left(\frac{1600+100}{12}\right)\times 10^{-6}$$

$$= 1.7\times 10^{-5}\ \text{A m}^2$$

$$\therefore B = \frac{4\times 1.7\times 10^{-5}}{3.4} = 2\times 10^{-5}\ \text{Wb m}^{-2}\ \text{or T}$$

Example 5.13: Two magnets with the same dimensions and mass, but of magnetic moments μ_1 = 100 A m^2 and μ_2 = 50 A m^2 are jointly suspended in the earth's magnetic field so as to perform angular oscillations in a horizontal plane. When their like poles are joined together, the period of their angular S.H.M. is 5 s. Find the period of angular S.H.M. when their unlike poles are joined together.

Solution:

$$T = 2\pi\sqrt{\frac{I}{\mu B}}$$

With like poles together, the effective magnetic moment is $(\mu_1 + \mu_2)$

$$\therefore T_1 = 2\pi\sqrt{\frac{I}{(\mu_1+\mu_2)B_H}}$$

With unlike poles together, the effective magnetic moment is $(\mu_1 - \mu_2)$

$$\therefore T_2 = 2\pi\sqrt{\frac{I}{(\mu_1-\mu_2)B_H}}$$

$$\therefore \frac{T_1}{T_2} = \sqrt{\frac{(\mu_1-\mu_2)}{(\mu_1+\mu_2)}}$$

$$\therefore \frac{5}{T_2} = \sqrt{\frac{1}{3}} \quad \therefore T_2 = \sqrt{75} = 8.660\ \text{s}$$

5.14 Damped Oscillations:

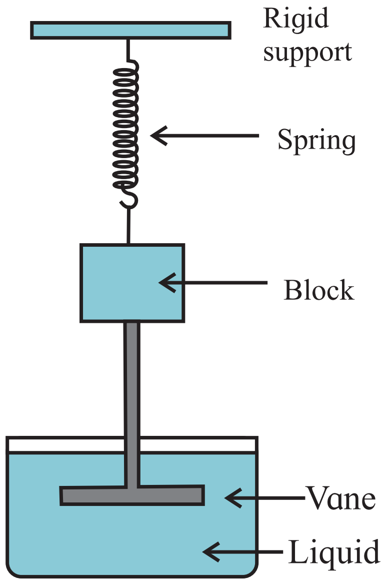


Fig. 5.13: A damped oscillator.

If the amplitude of oscillations of an oscillator is reduced by the application of an external force, the oscillator and its motion are said to be *damped*. *Periodic oscillations of gradually decreasing amplitude are called damped harmonic oscillations and the oscillator is called a damped harmonic oscillator*.

For example, the motion of a simple pendulum, dies eventually as air exerts a viscous force on the pendulum and there may be some friction at the support.

Figure 5.13 shows a block of mass m that can oscillate vertically on a spring. From the block, a rod extends to vane that is submerged on a liquid. As the vane moves up and down, the liquid exerts drag force on it, and thus on the complete oscillating system. The mechanical energy of the block-spring system decreases with time, as energy is transferred to thermal energy of the liquid and vane.

The damping force (F_d) depends on the nature of the surrounding medium and is directly proportional to the speed v of the vane and the block

$$\therefore F_d = -bv$$

Where b is the damping constant and negative sign indicates that F_d opposes the velocity.

For spring constant k, the force on the block from the spring is $F_s = -kx$.

Assuming that the gravitational force on the block is negligible compared to F_d and F_s, the total force acting on the mass at any time t is

$$F = F_d + F_s$$

$$\therefore ma = F_d + F_s$$

$$\therefore ma = -bv - kx$$

$$\therefore ma + bv + kx = 0$$

$$\therefore m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad \text{--- (5.35)}$$

The solution of Eq. (5.35) describes the motion of the block under the influence of a damping force which is proportional to the speed.

The solution is found to be of the form

$$x = Ae^{-bt/2m}\cos(\omega' t + \phi) \qquad \text{--- (5.36)}$$

$\left(Ae^{-bt/2m}\right)$ is the amplitude of the damped harmonic oscillations.

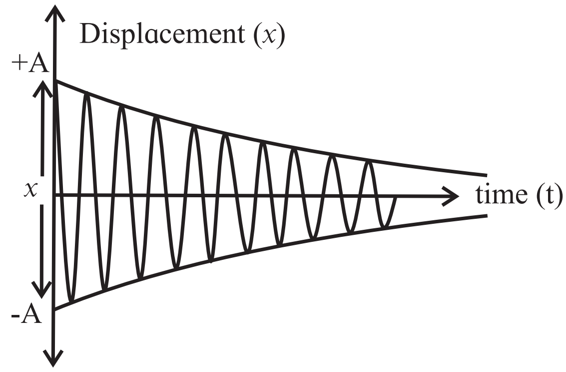


Fig. 5.14: Displacement against time graph.

As shown in the displacement against time graph (Fig 5.14), the amplitude decreases with time exponentially. The term $\cos(\omega' t + \phi)$ shows that the motion is still an S.H.M.

The angular frequency, $\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$

Period of oscillation, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}$

The damping increases the period (slows down the motion) and decreases the amplitude.

5.15 Free Oscillations, Forced Oscillations and Resonance:

Free Oscillations: If an object is allowed to oscillate or vibrate on its own, it does so with its natural frequency (or with one of its natural frequencies). For example, if the bob of a simple pendulum of length l is displaced and released, it will oscillate only with the frequency $n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ which is called its natural frequency and the oscillations are free oscillations. However, by applying a periodic force, the same pendulum can be made to oscillate with different frequency. The oscillations then will be forced oscillations and the frequency is driver frequency or forced frequency.

Consider the arrangement shown in the Fig. 5.15. There are four pendula tied to a string. Pendula A and C are of the same length, pendulum B is of shorter length and pendulum D is of longer length. Pendulum A has a solid rubber ball as its bob and acts as the driver pendulum or the source pendulum. Other three pendula have hollow rubber balls as their bobs and act as the driven pendula. As the pendula A and C are of the same lengths, their natural frequencies are the same. Pendulum B has higher natural frequency as its length is shorter than that of pendulum A. Natural frequency of pendulum D is less than that of pendula A and C.

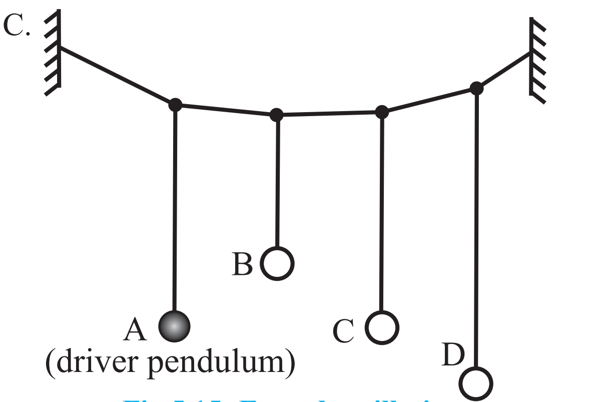


Fig 5.15: Forced oscillations.

Pendulum A is now set into oscillations in a plane perpendicular to the plane of paper. In the course of time it will be observed that the other three pendula also start oscillating in the same plane. This happens due to transfer of the vibrational energy through the string. Oscillations of pedulum A are free oscillations and those of pendula B, C and D are *forced oscillations* of the *same* frequency as that of A. The natural frequency of pendulum C is the same as that of A, as its length is the same as that of A.

It can also be seen that among the pendula B, C and D, the pendulum C oscillates with maximum amplitude and the other two with smaller amplitudes. As the energy depends upon the amplitude, it is clear that pendulum C has absorbed maximum energy from the source pendulum A, while the other two absorbed less. It shows that the object C having the same natural frequency as that of the source absorbs maximum energy from the source. In such case, it is said to be in *resonance* with the source (pendulum A). For unequal natural frequencies on either side (higher or lower), the energy absorbed (hence, the amplitude) is less. If the activity is repeated for a set of pendula of different lengths and squares of their amplitudes are plotted against their natural frequencies, the plot will be similar to that shown in the Fig. 5.16. The peak occurs when the forced frequency matches with the natural frequency, i.e., at the resonant frequency.

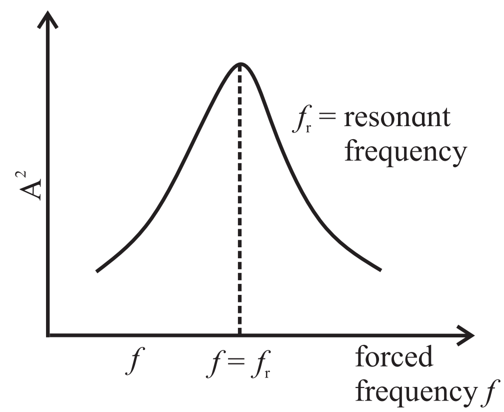


Fig 5.16: Resonant frequency.

In the next Chapter on superposition of waves, you will see that most of the traditional musical instruments use the principle of resonance. In the topic AC circuits, the resonance in the L.C. circuits is discussed.

Internet my friend

1. https://hyperphysics.phy-astr.gsu.edu/hbase/shm.html
2. https://hyperphysics.phy-astr.gsu.edu/hbase/pend.html
3. https://en.wikipedia.org/wiki/simpleharmonicmotion
4. https://opentextbc.ca/physicstextbook
5. https://physics.info

Exercises

1. Choose the correct option.

i) A particle performs linear S.H.M. starting from the mean position. Its amplitude is A and time period is T. At the instance when its speed is half the maximum speed, its displacement x is

(A) $\frac{\sqrt{3}}{2}A$ (B) $\frac{2}{\sqrt{3}}A$

(C) $A/2$ (D) $\frac{1}{\sqrt{2}}A$

ii) A body of mass 1 kg is performing linear S.H.M. Its displacement x (cm) at t (second) is given by $x = 6 \sin(100t + \pi/4)$. Maximum kinetic energy of the body is

(A) 36 J (B) 9 J

(C) 27 J (D) 18 J

iii) The length of second's pendulum on the surface of earth is nearly 1 m. Its length on the surface of moon should be [Given: acceleration due to gravity (g) on moon is 1/6 th of that on the earth's surface]

(A) 1/6 m (B) 6 m

(C) 1/36 m (D) $\frac{1}{\sqrt{6}}$ m

iv) Two identical springs of constant k are connected, first in series and then in parallel. A metal block of mass m is suspended from their combination. The ratio of their frequencies of vertical oscillations will be in a ratio

(A) 1:4 (B) 1:2 (C) 2:1 (D) 4:1

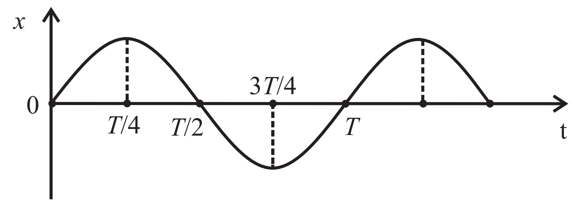
v) The graph shows variation of displacement of a particle performing S.H.M. with time t. Which of the following statements is correct from the graph?

(A) The acceleration is maximum at time T.

(B) The force is maximum at time $3T/4$.

(C) The velocity is zero at time $T/2$.

(D) The kinetic energy is equal to total energy at time $T/4$.



2. Answer in brief.

i) Define linear simple harmonic motion.

ii) Using differential equation of linear S.H.M, obtain the expression for (a) velocity in S.H.M., (b) acceleration in S.H.M.

iii) Obtain the expression for the period of a simple pendulum performing S.H.M.

iv) State the laws of simple pendulum.

v) Prove that under certain conditions a magnet vibrating in uniform magnetic field performs angular S.H.M.

3. Obtain the expression for the period of a magnet vibrating in a uniform magnetic field and performing S.H.M.

4. Show that a linear S.H.M. is the projection of a U.C.M. along any of its diameter.

5. Draw graphs of displacement, velocity and acceleration against phase angle, for a particle performing linear S.H.M. from (a) the mean position (b) the positive extreme position. Deduce your conclusions from the graph.

6. Deduce the expressions for the kinetic energy and potential energy of a particle executing S.H.M. Hence obtain the expression for total energy of a particle performing S.H.M and show that the total energy is conserved. State the factors on which total energy depends.

7. Deduce the expression for period of simple pendulum. Hence state the factors on which its period depends.

8. At what distance from the mean position is the speed of a particle performing S.H.M. half its maximum speed. Given path length of S.H.M. = 10 cm.

[Ans: 4.33 cm]

9. In SI units, the differential equation of an S.H.M. is $\frac{d^2x}{dt^2} = -36x$. Find its frequency and period.

[Ans: 0.9548 Hz, 1.047 s]

10. A needle of a sewing machine moves along a path of amplitude 4 cm with frequency 5 Hz. Find its acceleration $\left(\frac{1}{30}\right)$ s after it has crossed the mean position. [Ans: 34.2 m/s^2]

11. Potential energy of a particle performing linear S.H.M is 0.1 $\pi^2 x^2$ joule. If mass of the particle is 20 g, find the frequency of S.H.M. [Ans: 1.581 Hz]

12. The total energy of a body of mass 2 kg performing S.H.M. is 40 J. Find its speed while crossing the centre of the path.

[Ans: 6.324 cm/s]

13. A simple pendulum performs S.H.M of period 4 seconds. How much time after crossing the mean position, will the displacement of the bob be one third of its amplitude. [Ans: 0.2163 s]

14. A simple pendulum of length 100 cm performs S.H.M. Find the restoring force acting on its bob of mass 50 g when the displacement from the mean position is 3 cm. [Ans: 1.48 × 10^{-2} N]

15. Find the change in length of a second's pendulum, if the acceleration due to gravity at the place changes from 9.75 m/s^2 to 9.80 m/s^2.

[Ans: Decreases by 5.07 mm]

16. At what distance from the mean position is the kinetic energy of a particle performing S.H.M. of amplitude 8 cm, three times its potential energy?

[Ans: 4 cm]

17. A particle performing linear S.H.M. of period 2π seconds about the mean position O is observed to have a speed of $b\sqrt{3}\,\mathrm{m/s}$, when at a distance b (metre) from O. If the particle is moving away from O at that instant, find the time required by the particle, to travel a further distance b. [Ans: π/3 s]

18. The period of oscillation of a body of mass m_1 suspended from a light spring is T. When a body of mass m_2 is tied to the first body and the system is made to oscillate, the period is $2T$. Compare the masses m_1 and m_2 [Ans: 1/3]

19. The displacement of an oscillating particle is given by $x = a\sin\omega t + b\cos\omega t$ where a, b and ω are constants. Prove that the particle performs a linear S.H.M. with amplitude $A = \sqrt{a^2 + b^2}$

20. Two parallel S.H.M.s represented by x_1 = 5sin (4π t + π/3) cm and x_2 = 3sin (4πt + π/4) cm are superposed on a particle. Determine the amplitude and epoch of the resultant S.H.M.

[Ans: 7.936 cm, 54° 23']

21. A 20 cm wide thin circular disc of mass 200 g is suspended to a rigid support from a thin metallic string. By holding the rim of the disc, the string is twisted through 60° and released. It now performs angular oscillations of period 1 second. Calculate the maximum restoring torque generated in the string under undamped conditions. ($\pi^3 \approx 31$)

[Ans: 0.04133 N m]

22. Find the number of oscillations performed per minute by a magnet is vibrating in the plane of a uniform field of 1.6 × 10^{-5} Wb/m^2. The magnet has moment of inertia 3 × 10^{-6} kg m^2 and magnetic moment 3 A m^2.

[Ans:38.19 osc/min.]

23. A wooden block of mass m is kept on a piston that can perform vertical vibrations of adjustable frequency and amplitude. During vibrations, we don't want the block to leave the contact with the piston. How much maximum frequency is possible if the amplitude of vibrations is restricted to 25 cm? In this case, how much is the energy per unit mass of the block? (g ≈ π^2 ≈ 10 m s^{-2})

[Ans: n_{max} = 1/s, E/m = 1.25 J/kg]

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