3 Arithmetic Progression

Let's study.

- Sequence
- Arithmetic Progression
- n^{th} term of an A.P.
- Sum of n terms of an A.P.

Let's learn.

Sequence

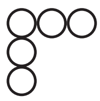
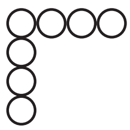
We write numbers 1, 2, 3, 4, . . . in an order. In this order we can tell the position of any number. For example, number 13 is at 13^{th} position. The numbers 1, 4, 9, 16, 25, 36, 49, . . . are also written in a particular order. Here $16 = 4^2$ is at 4^{th} position. similarly, $25 = 5^2$ is at the 5^{th} position; $49 = 7^2$ is at the 7^{th} position. In this set of numbers also, place of each number is detremined.

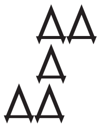
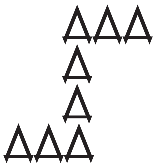
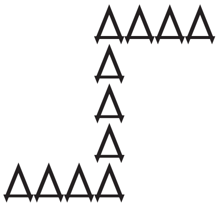
A set of numbers where the numbers are arranged in a definite order, like the natural numbers, is called a **sequence.**

In a sequence a particular number is written at a particular position. If the numbers are written as a_1, a_2, a_3, a_4 . . . then a_1 is first, a_2 is second, . . . and so on. It is clear that a_n is at the n^{th} place. A sequence of the numbers is also represented by alphabets f_1, f_2, f_3, . . . and we find that there is a definite order in which numbers are arranged.

When students stand in a row for drill on the playground they form a sequence. We have experienced that some sequences have a particular pattern.

Complete the given pattern

Pattern									
Number of circles	1	3	5	7					

Pattern							
Number of triangles	5	8	11				

Look at the patterns of the numbers. Try to find a rule to obtain the next number from its preceding number. This helps us to write all the next numbers.

See the numbers 2, 11, −6, 0, 5, −37, 8, 2, 61 written in this order.

Here $a_1 = 2$, $a_2 = 11$, $a_3 = -6$, . . . This list of numbers is also a sequence. But in this case we cannot tell why a particular term is at a particular position ; similarly we cannot tell a definite relation between the consecutive terms.

In general, only those sequences are studied where there is a rule which determines the next term.

For example (1) 4, 8, 12, 16 . . . (2) 2, 4, 8, 16, 32, . . .

(3) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}$ . . .

Terms in a sequence

In a sequence, ordered terms are represented as $t_1, t_2, t_3, \ldots \, . t_n$ . . . In general sequence is written as $\{t_n\}$. If the sequence is infinite, for every positive integer n, there is a term t_n.

Activity I : Some sequences are given below. Show the positions of the terms by $t_1, t_2, t_3, \ldots$

(1) 9, 15, 21, 27, . . . Here $t_1 = 9$, $t_2 = 15$, $t_3 = 21$, . . .

(2) 7, 7, 7, 7, . . . Here $t_1 = 7$, $t_2 = \square$, $t_3 = \square$, . . .

(3) −2, −6, −10, −14, . . . Here $t_1 = -2$, $t_2 = \square$, $t_3 = \square$, . . .

Activity II : Some sequences are given below. Check whether there is any rule among the terms. Find the similarity between two sequences.

To check the rule for the terms of the sequence look at the arrangements on the next page, and fill the empty boxes suitably.

(1) 1, 4, 7, 10, 13, . . . (2) 6, 12, 18, 24, . . .

(3) 3, 3, 3, 3, . . . (4) 4, 16, 64, . . .

(5) −1, −1.5, −2, −2.5, . . . (6) $1^3, 2^3, 3^3, 4^3$, . . .

Let's find the relation in these sequences. Let's understand the thought behind it.

(1) 1, 4, 7, 10, ☐, ☐, . . .

1 (+3), 4 (+3), 7 (+3), 10 (+3), ☐

(2) 6, 12, 18, 24, ☐, ☐, . . .

6 + 6, 12 + 6, 18 + 6, ☐, ☐

(3) 3, 3, 3, 3, ☐, ☐ . . .

3 + 0, 3 + 0, 3 + 0, ☐, ☐

(4) 4, 16, 64, ☐, ☐, ☐, . . .

4 × 4, 16 × 4, 64 × 4, ☐ × 4, ☐ × 4

(5) −1, −1.5, −2, −2.5, ☐ . . .

(−1)+(−0.5), −1.5 +(−0.5), −2 + (−0.5), −2.5 + (−0.5)

(6) 1^3, 2^3, 3^3, . . .

Here in the sequences (1), (2), (3), (5), the similarity is that next term is obtained by adding a particular number to the previous number. Each ot these sequences is called an Arithmetic Progression.

Sequence (4) is not an arithmetic progression. In this sequence the next term is obtained by mutliplying the previous term by a particular number. This type of sequences is called a Geometric Progression.

Sequence (6) is neither arithmetic progression nor geometric progression.

This year we are going to study arithmetic progression.

Arithmetic Progression

Some sequences are given below. For every sequence write the next three terms.

(1) 100, 70, 40, 10, . . . (2) −7, −4, −1, 2, . . . (3) 4, 4, 4, . . .

In the given sequences, observe how the next term is obtained.

1st term 2nd term 3rd term

(1) 100 70 40 10 -20 -50

100+(-30) 70+(-30) 40+(-30) 10+(-30) (-20)+(-30)

(2) -7 -4 -1 2 5 8

-7+3 -4+3 -1+3 2+3 5+3

(3) 4 4 4 4 4 . . . .

4 + 0 4 + 0 4 + 0 4 + 0

In each sequence above, every term is obtained by adding a particular number in the previous term. The difference between two consecutive terms is constant.

The diference in ex. (i) is negative, in ex. (ii) it is positive and in ex. (iii) it is zero.

If the difference between two consecutive terms is constant then it is called the common difference and is generally denoted by letter *d*.

In the given sequence if the difference between two consecutive terms $(t_{n+1} - t_n)$ is constant then the sequence is called Arithmetic Progression (A.P.). In this sequence $t_{n+1} - t_n = d$ is the common difference.

In an A.P. if first term is denoted by *a* and common difference is *d* then,

$t_1 = a$, $t_2 = a + d$

$t_3 = (a + d) + d = a + 2d$

A.P. having first term as *a* and common difference *d* is

$a, (a + d), (a + 2d), (a + 3d), \ldots\ldots$

Let's see some examples of A.P.

Ex.**(1)** Arifa saved ₹ 100 every month. In one year the total amount saved after every month is as given below.

Month	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Saving (’)	100	200	300	400	500	600	700	800	900	1000	1100	1200

The numbers showing the total saving after every month are in A.P.

Ex. **(2)** Pranav borrowed ₹ 10000 from his friend and agreed to repay ₹ 1000 per month. So the remaining amount to be paid in every month will be as follows.

No. of month	1	2	3	4	5	...	...	...
Amount to be paid (₹)	10,000	9,000	8,000	7,000	...	2,000	1,000	0

Ex. **(3)** Consider the table of 5, that is numbers divisible by 5.

5, 10, 15, 20, . . . 50, 55, 60, ... is an inifinite A.P.

Ex (1) and (2) are finite A.P. while (3) is an infinite A.P.

Let's remember!

(1) In a sequence if difference $(t_{n+1} - t_n)$ is constant then the sequence is called an arithmetic progression.

(2) In an A.P. the difference between two consecutive terms is constant and is denoted by d.

(3) Difference d can be positive, negative or zero.

(4) In an A.P. if the first term is a, and common difference is d then the terms in the sequence are a, $(a + d)$, $(a + 2d)$, . . .

Activity : Write one example of finite and infinite A.P. each.

Solved examples

Ex. **(1)** Which of the following sequences are A.P ? If it is an A.P, find next two terms.

(i) 5, 12, 19, 26, . . . (ii) 2, −2, −6, −10, . . .

(iii) 1, 1, 2, 2, 3, 3, . . . (iv) $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$

Solution : (i) In this sequence 5, 12, 19, 26, . . . ,

First term $= t_1 = 5$, $t_2 = 12$, $t_3 = 19$, . . .

$t_2 - t_1 = 12 - 5 = 7$

$t_3 - t_2 = 19 - 12 = 7$

Here first term is 5 and common difference which is constant is $d = 7$

$\therefore$ This sequence is an A.P.

Next two terms in this A.P. are $26 + 7 = 33$ and $33 + 7 = 40$.

Next two terms in given A.P. are 33 and 40

(ii) In the sequence 2, −2, −6, −10, . . . ,

$t_1 = 2$, $t_2 = -2$, $t_3 = -6$, $t_4 = -10$ . . .

$t_2 - t_1 = -2 - 2 = -4$

$t_3 - t_2 = -6 - (-2) = -6 + 2 = -4$

$t_4 - t_3 = -10 - (-6) = -10 + 6 = -4$

From this difference between two consecutive terms that is $t_n - t_{n-1} = -4$

$\therefore$ $d = -4$, which is constant. $\therefore$ It is an A.P.

Next two terms in this A.P. are $(-10) + (-4) = -14$ and $(-14) + (-4) = -18$

(iii) In the sequence 1, 1, 2, 2, 3, 3, . . . ,

$t_1 = 1$, $t_2 = 1$, $t_3 = 2$, $t_4 = 2$, $t_5 = 3$, $t_6 = 3$ . . .

$t_2 - t_1 = 1 - 1 = 0$ $t_3 - t_2 = 2 - 1 = 1$

$t_4 - t_3 = 2 - 2 = 0$ $t_3 - t_2 \neq t_2 - t_1$

In this sequence difference between two consecutive terms is not constant.

$\therefore$ This sequence is not an A.P.

(iv) In the sequence $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$,

$t_1 = \frac{3}{2}$, $t_2 = \frac{1}{2}$, $t_3 = -\frac{1}{2}$, $t_4 = -\frac{3}{2}$ . . .

$t_2 - t_1 = \frac{1}{2} - \frac{3}{2} = -\frac{2}{2} = -1$

$t_3 - t_2 = -\frac{1}{2} - \frac{1}{2} = -\frac{2}{2} = -1$

$t_4 - t_3 = -\frac{3}{2} - (-\frac{1}{2}) = -\frac{3}{2} + \frac{1}{2} = -\frac{2}{2} = -1$

Here the common difference $d = -1$ which is constant.

$\therefore$ Given sequence is an A.P. Let's find next two terms of this A.P.

$-\frac{3}{2} - 1 = -\frac{5}{2}$, $\frac{5}{2} - 1 = -\frac{7}{2}$

$\therefore$ Next two terms are $-\frac{5}{2}$ and $-\frac{7}{2}$

Ex. **(2)** The first term a and common difference d are given. Find first four terms of A.P.

(i) $a = -3, d = 4$ (ii) $a = 200, d = 7$

(iii) $a = -1, d = -\frac{1}{2}$ (iv) $a = 8, d = -5$

Solution : (i) Given $a = -3, d = 4$

$t_1 = -3$

$t_2 = t_1 + d = -3 + 4 = 1$

$t_3 = t_2 + d = 1 + 4 = 5$

$t_4 = t_3 + d = 5 + 4 = 9$

$\therefore$ A.P. is $= -3, 1, 5, 9, \ldots$

(ii) Given $a = 200, d = 7$

$a = t_1 = 200$

$t_2 = t_1 + d = 200 + 7 = 207$

$t_3 = t_2 + d = 207 + 7 = 214$

$t_4 = t_3 + d = 214 + 7 = 221$

$\therefore$ A.P. is $= 200, 207, 214, 221, \ldots$

(iii) $a = -1, d = -\frac{1}{2}$

$a = t_1 = -1$

$t_2 = t_1 + d = -1 + (-\frac{1}{2}) = -\frac{3}{2}$

$t_3 = t_2 + d = -\frac{3}{2} + (-\frac{1}{2}) = -\frac{4}{2} = -2$

$t_4 = t_3 + d = -2 + (-\frac{1}{2})$

$= -2 - \frac{1}{2} = -\frac{5}{2}$

$\therefore$ A.P. is $= -1, -\frac{3}{2}, -2, -\frac{5}{2}, \ldots$

(iv) $a = 8, d = -5$

$a = t_1 = 8$

$t_2 = t_1 + d = 8 + (-5) = 3$

$t_3 = t_2 + d = 3 + (-5) = -2$

$t_4 = t_3 + d = -2 + (-5) = -7$

$8, 3, -2, -7, \ldots$

$\therefore$ A.P. is $= 8, 3, -2, -7, \ldots$

Practice Set 3.1

1. Which of the following sequences are A.P. ? If they are A.P. find the common difference .

(1) $2, 4, 6, 8, \ldots$ (2) $2, \frac{5}{2}, 3, \frac{7}{3}, \ldots$ (3) $-10, -6, -2, 2, \ldots$

(4) $0.3, 0.33, .0333, \ldots$ (5) $0, -4, -8, -12, \ldots$ (6) $-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \ldots$

(7) $3, 3 + \sqrt{2}, 3 + 2\sqrt{2}, 3 + 3\sqrt{2}, \ldots$ (8) $127, 132, 137, \ldots$

2. Write an A.P. whose first term is a and common difference is d in each of the following.

(1) $a = 10, d = 5$ (2) $a = -3, d = 0$ (3) $a = -7, d = \frac{1}{2}$

(4) $a = -1.25, d = 3$ (5) $a = 6, d = -3$ (6) $a = -19, d = -4$

3. Find the first term and common difference for each of the A.P.

(1) 5, 1, –3, –7, . . . (2) 0.6, 0.9, 1.2, 1.5, . . .

(3) 127, 135, 143, 151, . . . (4) $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \ldots$

Let's think.

- Is 5, 8, 11, 14, . . . an A.P. ? If so then what will be the 100th term? Check whether 92 is in this A.P.? Is number 61 in this A.P.?

Let's learn.

n^{th} term of an A. P.

In the sequence 5, 8, 11, 14, . . . the difference between two consecutive terms is 3. Hence, this sequence is an A.P.

Here the first term is 5. If 3 is added to 5 we get the second term 8. Similarly to find 100th term what should be done?

	First term	Second term	Third term	. . .
Number	5,	5 + 3 = 8	8 + 3 = 11	. . .

In this way reaching upto 100th term will be time consuming. Let's see if we can find any formula for it.

5	8	11	14	. . .	. . .	. . .	. . .
5	$5 + 1 \times 3$	$5 + 2 \times 3$	$5 + 3 \times 3$	. . .	$5 + (n - 1) \times 3$	$5 + n \times 3$	. . .
1st term t_1	2nd term t_2	3rd term t_3	4th term t_4	. . .	n^{th} term t_n	$(n + 1)^{th}$ term t_{n+1}	. . .

Generally in the A.P. $t_1, t_2, t_3, \ldots$ If first term is a and common difference is d,

$$t_1 = a$$

$$t_2 = t_1 + d = a + d = a + (2 - 1)\, d$$

$$t_3 = t_2 + d = a + d + d = a + 2d = a + (3 - 1)d$$

$$t_4 = t_3 + d = a + 2d + d = a + 3d = a + (4 - 1)d$$

We get $t_n = a + (n - 1)\, d$.

Using the above formula we can find the 100^{th} term of the A.P. 5, 8, 11, 14, . . .

Here $a = 5$ $d = 3$

$$t_n = a + (n - 1)d$$

$$\therefore \ t_{100} = 5 + (100 - 1) \times 3$$

$$= 5 + 99 \times 3$$

$$= 5 + 297$$

$$t_{100} = 302$$

100^{th} tem of this A.P. is 302.

Let's check whether 61 is in this A.P. To find the answer we use the same formula.

$$t_n = a + (n - 1)d$$

$$t_n = 5 + (n - 1) \times 3$$

If 61 is n^{th} term means t_n, then

$$\therefore \ 61 = 5 + 3n - 3$$

$$= 3n + 2$$

$$\therefore \ 3n = 59$$

$$\therefore \ n = \frac{59}{3}$$

But then, n is not a natural number.

$\therefore$ 61 is not in this A.P.

Let's think.

Kabir's mother keeps a record of his height on each birthday. When he was one year old, his height was 70 cm, at 2 years he was 80 cm tall and 3 years he was 90 cm tall. His aunt Meera was studying in the 10^{th} class. She said, ''it seems like Kabir's height grows in Arithmetic Progression''. Assuming this, she calculated how tall Kabir will be at the age of 15 years when he is in the 10^{th} ! She was shocked to find it. You too assume that Kabir grows in A.P. and find out his height at the age of 15 years.

Solved examples

Ex. (1) Find t_n for following A.P. and then find 30^{th} term of A.P.

3, 8, 13, 18, . . .

Solution :Given A.P. 3, 8, 13, 18, . . .

Here $t_1 = 3$, $t_2 = 8$, $t_3 = 13$, $t_4 = 18$, . . .

$d = t_2 - t_1 = 8 - 3 = 5$

We know that $t_n = a + (n - 1)d$

$\therefore t_n = 3 + (n - 1) \times 5 \quad \because a = 3, d = 5$

$\therefore t_n = 3 + 5n - 5$

$\therefore t_n = 5n - 2$

$\therefore$ 30^{th} term $= t_{30} = 5 \times 30 - 2$

$= 150 - 2 \quad = 148$

Ex. (2) Which term of the following A.P. is 560 ?

2, 11, 20, 29, . . .

Solution : Given A.P. 2, 11, 20, 29, . .

Here $a = 2$, $d = 11 - 2 = 9$

n^{th} term of this A.P. is 560.

$t_n = a + (n - 1)d$

$\therefore 560 = 2 + (n - 1) \times 9$

$= 2 + 9n - 9$

$\therefore 9n = 567$

$\therefore n = \frac{567}{9} = 63$

$\therefore$ 63^{rd} term of given A.P. is 560.

Ex. (3) Check whether 301 is in the sequence 5, 11, 17, 23, . . . ?

Solution : In the sequence 5, 11, 17, 23,. . .

$t_1 = 5$, $t_2 = 11$, $t_3 = 17$, $t_4 = 23$, . . .

$t_2 - t_1 = 11 - 5 = 6$

$t_3 - t_2 = 17 - 11 = 6$

$\therefore$ This sequence is an A.P.

First term $a = 5$ and $d = 6$

If 301 is n^{th} term, then.

$t_n = a + (n - 1)d = 301$

$\therefore 301 = 5 + (n - 1) \times 6$

$= 5 + 6n - 6$

$\therefore 6n = 301 + 1 = 302$

$\therefore n = \frac{302}{6}$. But it is not an integer.

$\therefore$ 301 is not in the given sequence.

Ex. **(4)** How many two digit numbers are divisible by 4 ?

Solution : List of two digit numbers divisible by 4 is

12, 16, 20, 24, . . . , 96.

Let's find how many such numbers are there.

$t_n = 96$, $a = 12$, $d = 4$

From this we will find the value of n.

$t_n = 96$, $\therefore$ By formula,

$96 = 12 + (n - 1) \times 4$

$= 12 + 4n - 4$

$\therefore 4n = 88$

$\therefore n = 22$

$\therefore$ There are 22 two digit numbers divisible by 4.

Ex. (5) – The 10th term and the 18th term of an A.P. are 25 and 41 respectively then find 38th term of that A.P., similarly if n^{th} term is 99. Find the value of n.

Solution : In the given A.P. $t_{10} = 25$ and $t_{18} = 41$.

We know that, $t_n = a + (n-1)d$

$\therefore \quad t_{10} = a + (10-1)\,d$

$\therefore \quad 25 = a + 9d$ . . . (I)

Similarly $t_{18} = a + (18-1)\,d$

$\therefore \quad 41 = a + 17d$ . . . (II)

$25 = a + 9d$ . . . From (I) .

$a = 25 - 9d.$

Substituting this value in equation II.

$\therefore$ Equation (II) $a + 17d = 41$

$\therefore \quad 25 - 9d + 17d = 41$

$8d = 41 - 25 = 16$

$d = 2$

Substituting $d = 2$ in equation I.

$a + 9d = 25$

$\therefore\ a + 9 \times 2 = 25$

$\therefore\ a + 18 = 25$

$\therefore\ a = 7$

Now, $t_n = a + (n-1)d$

$\therefore \quad t_{38} = 7 + (38-1) \times 2$

$t_{38} = 7 + 37 \times 2$

$t_{38} = 7 + 74$

$t_{38} = 81$

If n^{th} term is 99, then to find value of n.

$t_n = a + (n-1)d$

$99 = 7 + (n-1) \times 2$

$99 = 7 + 2n - 2$

$99 = 5 + 2n$

$\therefore\ 2n = 94$

$\therefore\ n = 47$

$\therefore$ In the given progression 38th term is 81 and 99 is the 47th term.

Practice Set 3.2

1. Write the correct number in the given boxes from the following A. P.

(i) 1, 8, 15, 22, . . .

Here $a = \square$, $t_1 = \square$, $t_2 = \square$, $t_3 = \square$,

$t_2 - t_1 = \square - \square = \square$

$t_3 - t_2 = \square - \square = \square$ $\therefore d = \square$

(ii) 3, 6, 9, 12, . . .

Here $t_1 = \square$, $t_2 = \square$, $t_3 = \square$, $t_4 = \square$,

$t_2 - t_1 = \square$, $t_3 - t_2 = \square$ $\therefore d = \square$

(iii) −3, −8, −13, −18, . . .

Here $t_3 = \square$, $t_2 = \square$, $t_4 = \square$, $t_1 = \square$,

$t_2 - t_1 = \square$, $t_3 - t_2 = \square$ $\therefore a = \square$, $d = \square$

(iv) 70, 60, 50, 40, . . .

Here $t_1 = \square$, $t_2 = \square$, $t_3 = \square$, . . .

$\therefore a = \square$, $d = \square$

2. Decide whether following sequence is an A.P., if so find the 20th term of the progression.

−12, −5, 2, 9, 16, 23, 30, . . .

3. Given Arithmetic Progression 12, 16, 20, 24, . . . Find the 24th term of this progression.

4. Find the 19th term of the following A.P.

7, 13, 19, 25, . . .

5. Find the 27th term of the following A.P.

9, 4, −1, −6, −11, . . .

6. Find how many three digit natural numbers are divisible by 5.

7. The 11th term and the 21st term of an A.P. are 16 and 29 respectively, then find the 41th term of that A.P.

8. 11, 8, 5, 2, . . . In this A.P. which term is number −151?

9. In the natural numbers from 10 to 250, how many are divisible by 4?

10. In an A.P. 17th term is 7 more than its 10th term. Find the common difference.

The Wise Teacher

Once upon a time, there lived a king. He appointed two teachers Tara and Meera to teach horse riding for one year to his children Yashwantraje and Geetadevi. He asked both of them how much salary they wanted.

Tara said, ''Give me 100 gold coins in first month and every month increase the amount by 100 gold coins.'' Meera said, ''Give me 10 gold coins in the first month and every month just double the amount of the previous month.''

The king agreed. After three months Yashwantraje said to his sister, ''My teacher is smarter than your teacher as she had asked for more money.'' Geetadevi said, ''I also thought the same, I asked Meeratai about it. She only smiled and said compare the salaries after 8 months. I calculated their 9 months salaries. You can also check.''

Months	1	2	3	4	5	6	7	8	9	10	11	12
Tara's salary	100	200	300	400	500	600	700	800	900	-	-	-
Meera's salary	10	20	40	80	160	320	640	1280	2560	-	-	-

Complete the above table.

Tara's salary 100, 200, 300, 400, . . . is in A.P.

$t_1 = 100,\ \ t_2 = 200,\ t_3 = 300, . . . \qquad t_2 - t_1 = 100 = d$

Common difference is 100.

Meera's salary 10, 20, 40, 80, . . . is not an A.P. Reason is 20 − 10 = 10, 40 − 20 = 20, 80 − 40 = 40 So the common difference is not constant.

But here each term is double the preceding term.

Here $\frac{t_2}{t_1} = \frac{20}{10} = 2,\ \frac{t_3}{t_2} = \frac{40}{20} = 2,\ \frac{t_4}{t_3} = \frac{80}{40} = 2$

$\therefore \frac{t_{n+1}}{t_n}$ The ratio of a term and its preceding term is constant. This type of progression is called a geometric progression. Notice that if ratio $\frac{t_{n+1}}{t_n}$ is greater than 1, then geometric progression will increase faster than arithmetic progression.

If the ratio is smaller than 1, note how the geometric progression changes.

This year we are going to study Arithmetic Progression only. We have seen how to find the n^{th} term of an A.P. Now we are going to see how to find the sum of the first n terms.

Quick Addition

Three hundred years ago there was a single teacher school in Germany. The teacher was Buttner and he had an assistant Johann Martin Bortels. He used to teach alphabets to the children and sharpen their pencils. Buttner was a strict teacher. One day he wanted to do some work and wanted peace in the class, so he tried to occupy all students with a lengthy addition. They were asked to add all intergers from 1 to 100. In few minutes one slate was slammed on the floor. He looked at Carl Gauss and asked, ''I asked you to add all integers from 1 to 100. Why did you keep the slate down? Don't you want to do it ?''

Carl Gauss said, ''I have done the addition.''

The teacher asked, ''How did you do it so quickly? You wouldn't have written all the numbers ! What is the answer ?''

Carl Gauss said, ''Five thousand fifty''

Teacher was so surprised and asked him, 'How do you find the answer?''

Carl Gauss explained his quick addition method:

Nos. in increasing order	1	2	3	4	. . . .	100
	+	+	+	+		+
Nos. in decreasing order	100	99	98	97	. . . .	1
Sum	101	101	101	101		101

The sum of each pair is 101. This sum occurs 100 times so 101×100 is the product needed. It is 10100. In this 1 to 100 are counted two times. Therefore, half of 10100 is 5050 and sum of 1 to 100 is 5050. The teacher appreciated his work.

Now using this method of Gauss, let's find sum of n terms of an A.P.

Johann Friedrich Carl Gauss

30th April 1777 – 23rd February 1855.

Carl Gauss was a great German mathematician, He was born in Braunschweig, he was the only son of uneducated parents. He showed a glimpse of his intelligence in Buttner's school. After some years, Buttner's helper, Johann Martin Bartels and Gauss became friends. Together, they published a book on Algebra. Bartels made the other people realise the extra ordinary intelligence of Gauss.

Let's learn.

Sum of first n terms of an A. P.

Arithmetic Progression $a, a + d, a + 2d, a + 3d, \ldots\ a + (n - 1)d$

In this progression a is the first term and d is the common difference. Let's write the sum of first n terms as S_n.

$$S_n = [a] + [a + d] + \ldots + [a+(n-2)d] + [a+(n-1)d]$$

Reversing the terms and rewritting the expression again,

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \ldots + [a + d] + [a]$$

On adding,

$$2S_n = [a+a+(n-1)d] + [a + d+a+(n-2)d] + \ldots + [a+(n-2)d+ a + d] + [a+(n-1)d+a]$$

$$2S_n = [2a+(n-1)d] + [2a+(n-1)d] + \ldots + [2a+(n-1)d] \ldots n \text{ times.}$$

$$\therefore\ 2S_n = n\,[2a+(n-1)d]$$

$$\therefore\ S_n = \frac{n}{2}[2a+(n-1)d] \quad \text{or} \quad S_n = na + \frac{n(n-1)}{2}d$$

Ex. Let's find the sum of first 100 terms of A.P. 14, 16, 18, ...

Here $a = 14$, $d = 2$, $n = 100$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$\therefore\ S_{100} = \frac{100}{2}[2 \times 14+(100-1) \times 2]$$

$$= 50\,[28 + 198]$$

$$= 50 \times 226 = 11300$$

$\therefore$ Sum of first 100 terms of given A.P. is 11,300

Let's remember!

For the given Arithmetic Progression, if first term is a and common diffrence is d then

$$t_n = [a+(n-1)d]$$

$$S_n = \frac{n}{2}[2a+(n-1)d] = na + \frac{n(n-1)}{2}d$$

Let's find one more formula for sum of first n terms.

In the A.P. $a, a+d, a+2d, a+3d, \ldots\ a+(n-1)d$

First term = $t_1 = a$ and n^{th} term is $[a+(n-1)d]$

Now $S_n = \frac{n}{2}\left[a+a+(n-1)d\right]$

$\therefore\ S_n = \frac{n}{2}[t_1 + t_n] = \frac{n}{2}$ [First term+ last term]

Solved examples

Ex. (1) Find the sum of first n natural numbers.

Solution : First n natural numbers are $1, 2, 3, \ldots, n$.

Here $a = 1,\ d = 1,\ n^{th}$ term $= n$

$\therefore\ S_n = 1 + 2 + 3 + \ldots + n$

$S_n = \frac{n}{2}$ [First term+ last term] . . . . . (by the formula)

$= \frac{n}{2}[1+n]$

$= \frac{n(n+1)}{2}$

$\therefore$ Sum of first n natural number is $\frac{n(n+1)}{2}$.

Ex. (2) Find the sum of first n even natural numbers.

Solution : First n even natural numbers are $2, 4, 6, 8, \ldots, 2n$.

t_1 = First term = 2 , t_n = last term = $2n$

Method I

$= \frac{n}{2}[t_1 + t_n]$

$= \frac{n}{2}[2+2n]$

$= \frac{n}{2} \times 2\,(1+n)$

$= n\,(1+n)$

Method II

$S_n = 2 + 4 + 6 \ldots + 2n$

$= 2(1 + 2 + 3 + \ldots + n)$

$= \frac{2[n(n+1)]}{2}$

$= n\,(1+n)$

Method III

$S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2 \times 2+(n-1)2]$

$= \frac{n}{2}[4+2n-2]$

$= \frac{n}{2}[2+2n]$

$= \frac{n}{2} \times 2\,(1+n)$

$= n\,(1+n)$

$\therefore$ Sum of first n even natural numbers is $n\,(1+n)$.

Ex. (3) Find the sum of first n odd natural numbers.

Solution : First n natural numbers

$1, 3, 5, 7, \ldots, (2n - 1)$.

$a = t_1 = 1$ and $t_n = (2n - 1)$, $d = 2$

Method I

$S_n = \frac{n}{2}[t_1 + t_n]$

$= \frac{n}{2}[1 + (2n - 1)]$

$= \frac{n}{2}[1 + 2n - 1]$

$= \frac{n}{2} \times 2n$

$= n^2$

Method II

$S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2 \times 1 + (n-1) \times 2]$

$= \frac{n}{2}[2 + 2n - 2]$

$= \frac{n}{2} \times 2n$

$= n^2$

Method III

$1 + 3 + \ldots 2n-1$

$= (1 + 2 + 3 + \ldots + 2n)$

$- (2 + 4 + 6 + \ldots + 2n)$

$= \frac{2n(2n+1)}{2} - \frac{2n(n+1)}{2}$

$= (2n^2 + n) - (n^2 + n)$

$= n^2$

Ex. (4) Find the sum of all odd numbers from 1 to 150.

Solution : 1 to 150 all odd numbers are $1, 3, 5, 7, \ldots, 149$.

Which is an A.P.

Here $a = 1$ and $d = 2$. First let's find how many odd numbers are there from 1 to 150, so find the value of n, if $t_n = 149$

$t_n = a + (n - 1)d$

$149 = 1 + (n - 1)2$ $\qquad \therefore 149 = 1 + 2n - 2$

$\therefore n = 75$

Now let's find the sum of these 75 numbers $1 + 3 + 5 + \ldots + 149$.

$a = 1$ and $d = 2$, $n = 75$

Method I $S_n = \frac{n}{2}[2a+(n-1)d]$

$S_n = \boxed{}$

$S_n = \boxed{} \times \boxed{}$

$S_n = \boxed{}$

Method II $S_n = \frac{n}{2}[t_1 + t_n]$

$S_n = \frac{75}{2}[1 + 149]$

$S_n = \boxed{} \times \boxed{}$

$S_n = \boxed{}$

Practice Set 3.3

1. First term and common difference of an A.P. are 6 and 3 respectively ; find S_{27}.

$a = 6,\ d = 3,\ S_{27} = ?$

$S_n = \frac{n}{2}[\square + (n-1)d]$

$S_{27} = \frac{27}{2}[12 + (27-1)\square]$

$= \frac{27}{2} \times \square$

$= 27 \times 45 \quad = \square$

2. Find the sum of first 123 even natural numbers.

3. Find the sum of all even numbers between 1 and 350.

4. In an A.P. 19^{th} term is 52 and 38^{th} term is 128, find sum of first 56 terms.

5. Complete the following activity to find the sum of natural numbers between 1 and 140 which are divisible by 4.

Between 1 and 140, natural numbers divisible by 4

↓

4, 8, ..., 136

↓

How many numbers ? ∴ $n = \square$

↓

$n = \square$, $a = \square$, $d = \square$

↓

$t_n = a+(n-1)d$

↓

$136 = \square + (n - 1) \times \square$

→ $n = \square$ → $S_n = \frac{n}{2}[2a+(n-1)d]$

$S_{\square} = \frac{\square}{2}[\qquad\qquad]$ $= \square$

Sum of numbers from 1 to 140, which are divisible by 4 = $\square$

6.★ Sum of first 55 terms in an A.P. is 3300, find its 28^{th} term.

7.★ In an A.P. sum of three consecutive terms is 27 and their product is 504, find the terms. (Assume that three consecutive terms in A.P. are $a - d$, a, $a + d$.)

8.★ Find four consecutive terms in an A.P. whose sum is 12 and sum of 3rd and 4th term is 14.

(Assume the four consecutive terms in A.P. are $a - d$, a, $a + d$, $a + 2d$.)

9.★ If the 9th term of an A.P. is zero then show that the 29th term is twice the 19th term.

Let's learn.

Application of A.P.

Ex. (1) A mixer manufacturing company manufactured 600 mixers in 3rd year and in 7th year they manufactured 700 mixers. If every year there is same growth in the production of mixers then find (i) Production in the first year (ii) Production in 10th year (iii) Total production in first seven years.

Solution : Addition in the number of mixers manufactured by the company per year is constant therefore the number of production in successive years is in A.P. (i) Let's assume that company manufactured t_n mixers in the n^{th} year then as per given information,

$t_3 = 600$, $t_7 = 700$

We know that $t_n = a+(n-1)d$

$t_3 = a+(3-1)d$

$a + 2d = 600. . .$ (I)

$t_7 = a+(7-1)d$

$t_7 = a+6d = 700$

$a+2d = 600$ $\therefore$ Substituting $a = 600 - 2d$ in equation (II),

$600 - 2d + 6d = 700$

$4d = 100 \quad \therefore d = 25$

$a+2d = 600 \quad \therefore a + 2 \times 25 = 600$

$a + 50 = 600 \quad \therefore a = 550$

$\therefore$ Production in first year was 550.

(ii) $t_n = a+(n-1)d$

$t_{10} = 550+(10-1) \times 25$

$= 550 + 225$

Production in 10th year was 775.

(iii) For finding total production in first 7 years let's use formula for S_n.

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{7}{2}[1100 + 150] = \frac{7}{2}[1250] = 7 \times 625 = 4375$$

Total production in first 7 years is 4375 mixers.

Ex. (2) Ajay sharma repays the borrowed amount of ₹ 3,25,000 by paying ₹ 30500 in the first month and then decreases the payment by ₹ 1500 every month. How long will it take to clear his amount?

Solution : Let the time required to clear the amount be n months. The monthly payment decreases by ₹ 1500. Therefore the payments are in A.P.

First term = a = 30500, d = -1500

Amount = S_n = 3,25,000

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$3{,}25{,}000 = \frac{n}{2}[2 \times 30500+(n-1)d]$$

$$= \frac{n}{2}[2 \times 30500 - 1500n + 1500]$$

$$3{,}25{,}000 = 30500n - 750n^2 + 750n$$

$$\therefore 750n^2 - 31250n + 325000 = 0$$

divide both sides by 250.

$$\therefore 3n^2 - 125n + 1300 = 0$$

$$\therefore 3n^2 - 60n - 65n + 1300 = 0$$

$$\therefore 3n(n-20) - 65(n-20) = 0$$

$$\therefore (n - 20)(3n - 65) = 0$$

$$\therefore n - 20 = 0 \text{ , } 3n - 65 = 0$$

$$\therefore n = 20 \text{ or } n = \frac{65}{3} = 21\frac{2}{3}$$

In an A.P. n is a natural number.

$$\therefore \quad n \neq \frac{65}{3} \qquad \therefore \quad n = 20$$

(Or, after 20 months, S_{20} = 3,25,000 then the total amount will be repaid. It is not required to think about further period of time.)

$\therefore$ To clear the amount 20 months are needed.

Ex. (3) Anvar saves some amount every month. In first three months he saves ₹ 200, ₹ 250 and ₹ 300 respectively. In which month will he save ₹ 1000?

Solution: Saving in first month ₹ 200; Saving in second month ₹ 250; .....

200, 250, 300, . . . this is an A.P.

Here a = 200, d = 50, Let's find n using t_n formula and then find S_n.

$$t_n = a+(n-1)d$$
$$= 200 + (n-1)50$$
$$= 200 + 50n - 50$$
$$1000 = 150 + 50n$$
$$150 + 50n = 1000$$
$$50n = 1000 - 150$$
$$50n = 850$$
$$\therefore\ n = 17$$

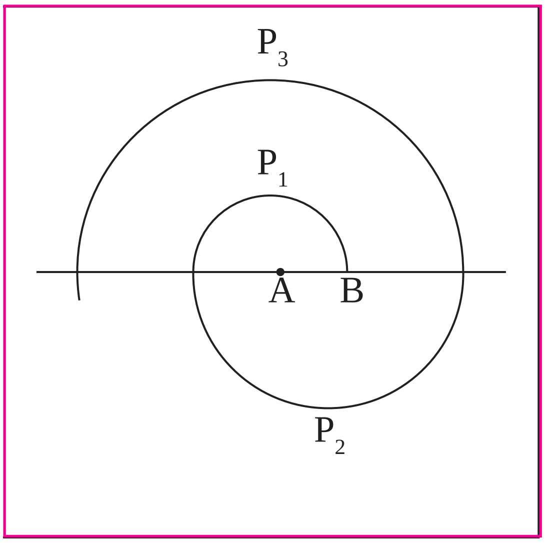
In the 17th month he will save ₹ 1000.

Let's find that in 17 months how much total amount is saved.

$$S_n = \frac{n}{2}\,[2a+(n-1)d]$$
$$= \frac{17}{2}\,[2 \times 200+(17-1) \times 50]$$
$$= \frac{17}{2}\,[400 + 800]$$
$$= \frac{17}{2}\,[1200]$$
$$= 17 \times 600$$
$$= 10200$$

In 17 months total saving is ₹ 10200.

Ex. (4) As shown in the figure, take point A on the line and draw a half circle P_1 of radius 0.5 with A as centre. It intersects given line in point B. Now taking B as centre draw a half circle P_2 of radius 1 cm which is on the other side of the line.



Now again taking A as centre draw a half circle P_3 of radius 1.5 cm. If we draw half circles like this having radius 0.5 cm, 1 cm, 1.5 cm, 2 cm, we get a figure of spiral shape. Find the length of such spiral shaped figure formed by 13 such half circles. ($\pi = \frac{22}{7}$)

Solution : Semi circumferences $P_1, P_2, P_3, \ldots$ are drawn by taking centres A, B, A, B,... It is given that radius of the first circle is 0.5 cm. The radius of the second circle is 1.0 cm,... From this information we will find $P_1, P_2, P_3, \ldots P_3$.

Length of the first semi circumference = $P_1 = \pi r_1 = \pi \times \frac{1}{2} = \frac{\pi}{2}$

$$P_2 = \pi r_2 = \pi \times 1 = \pi$$

$$P_3 = \pi r_3 = \pi \times 1.5 = \frac{3}{2}\pi$$

The lengths are $P_1, P_2, P_3, \ldots$, and the numbers $\frac{1}{2}\pi, 1\pi, \frac{3}{2}\pi, \ldots$ are in A.P.

Here $a = \frac{1}{2}\pi$, $d = \frac{1}{2}\pi$, From this let's find S_{13}.

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{13} = \frac{13}{2}\left[2 \times \frac{\pi}{2} + (13-1) \times \frac{1}{2}\pi\right]$$

$$= \frac{13}{2}[\pi + 6\pi]$$

$$= \frac{13}{2} \times 7\pi =$$

$$= \frac{13}{2} \times 7 \times \frac{22}{7}$$

$$= 143 \text{ cm.}$$

$\therefore$ The total length of spiral shape formed by 13 semicircles is 143 cm.

Ex. **(5)** In the year 2010 in the village there were 4000 people who were literate. Every year the number of literate people increases by 400. How many people will be literate in the year 2020?

Solution :

Year	2010	2011	2012	...	2020
Literate People	4000	4400	4800	...	☐

$a = 4000, \quad d = 400 \quad n = 11$

$t_n = a+(n-1)d$

$= 4000 + (11-1)400$

$= 4000 + 4000$

$= 8000$

In year 2020, 8000 people will be literate.

Ex. **(6)** In year 2015, Mrs. Shaikh got a job with salary ₹ 1,80,000 per year. Her employer agreed to give ₹ 10,000 per year as increment. Then in how many years will her annual salary be ₹ 2,50,000?

Solution :

Year	First Year (2015)	Second Year (2016)	Third Year (2017)	...
Salary (₹)	[1,80,000]	[1,80,000 + 10,000]		...

$a = 1,80,000 \qquad d = 10,000 \qquad n = ? \qquad t_n = 2,50,000$ ₹

$t_n = a+(n-1)d$

$2,50,000 = 1,80,000 + (n-1) \times 10,000$

$(n-1) \times 10000 = 70,000$

$(n-1) = 7$

$n = 8$

In the 8th year her annual salary will be ₹ 2,50,000.

Practice Set 3.4

1. On 1st Jan 2016, Sanika decides to save ₹ 10, ₹ 11 on second day, ₹ 12 on third day. If she decides to save like this, then on 31st Dec 2016 what would be her total saving ?
2. A man borrows ₹ 8000 and agrees to repay with a total interest of ₹ 1360 in 12 monthly instalments. Each instalment being less than the preceding one by ₹ 40. Find the amount of the first and last instalment.
3. Sachin invested in a national saving certificate scheme. In the first year he invested ₹ 5000, in the second year ₹ 7000, in the third year ₹ 9000 and so on. Find the total amount that he invested in 12 years.
4. There is an auditorium with 27 rows of seats. There are 20 seats in the first row, 22 seats in the second row, 24 seats in the third row and so on. Find the number of seats in the 15th row and also find how many total seats are there in the auditorium ?
5. Kargil's temperature was recorded in a week from Monday to Saturday. All readings were in A.P.The sum of temperatures of Monday and Saturday was 5° C more than sum of temperatures of Tuesday and Saturday. If temperature of Wednesday was -30° celsius then find the temperature on the other five days.
6. On the world environment day tree plantation programme was arranged on a land which is triangular in shape. Trees are planted such that in the first row there is one tree, in the second row there are two trees, in the third row three trees and so on. Find the total number of trees in the 25 rows.

Problem Set - 3

1. Choose the correct alternative answer for each of the following sub questions.

(1) The sequence $-10, -6, -2, 2, \ldots$

(A) is an A.P., Reason $d = -16$ (B) is an A.P., Reason $d = 4$

(C) is an A.P., Reason $d = -4$ (D) is not an A.P.

(2) First four terms of an A.P. are ....., whose first term is -2 and common difference is -2.

(A) $-2, 0, 2, 4$ (B) $-2, 4, -8, 16$

(C) $-2, -4, -6, -8$ (D) $-2, -4, -8, -16$

(3) What is the sum of the first 30 natural numbers ?

(A) 464 (B) 465 (C) 462 (D) 461

(4) For an given A.P. $t_7 = 4$, $d = -4$ then $a = \ldots$

(A) 6 (B) 7 (C) 20 (D) 28

(5) For an given A.P. $a = 3.5$, $d = 0$, $n = 101$, then $t_n = \ldots$

(A) 0 (B) 3.5 (C) 103.5 (D) 104.5

(6) In an A.P. first two terms are -3, 4 then 21st term is . . .

(A) -143 (B) 143 (C) 137 (D) 17

(7) If for any A.P. $d = 5$ then $t_{18} - t_{13} = \ldots$

(A) 5 (B) 20 (C) 25 (D) 30

(8) Sum of first five multuiples of 3 is. . .

(A) 45 (B) 55 (C) 15 (D) 75

(9) 15, 10, 5, . . . In this A.P. sum of first 10 terms is . . .

(A) -75 (B) -125 (C) 75 (D) 125

(10) In an A.P. 1st term is 1 and the last term is 20. The sum of all terms is = 399 then $n = \ldots$

(A) 42 (B) 38 (C) 21 (D) 19

2. Find the fourth term from the end in an A.P. $-11, -8, -5, \ldots, 49$.

3. In an A.P. the 10th term is 46, sum of the 5th and 7th term is 52. Find the A.P.

4. The A.P. in which 4th term is -15 and 9th term is -30. Find the sum of the first 10 numbers.

5. Two A.P.'s are given 9, 7, 5, . . . and 24, 21, 18, ... If n^{th} term of both the progressions are equal then find the value of n and n^{th} term.

6. If sum of 3rd and 8th terms of an A.P. is 7 and sum of 7th and 14th terms is -3 then find the 10th term.

7. In an A.P. the first term is -5 and last term is 45. If sum of all numbers in the A.P. is 120, then how many terms are there? What is the common difference?

8. Sum of 1 to n natural numbers is 36, then find the value of n.

9. Divide 207 in three parts, such that all parts are in A.P. and product of two smaller parts will be 4623.

10. There are 37 terms in an A.P., the sum of three terms placed exactly at the middle is 225 and the sum of last three terms is 429. Write the A.P.

11★ If first term of an A.P. is a, second term is b and last term is c, then show that sum of all terms is $\frac{(a+c)(b+c-2a)}{2(b-a)}$.

12.★ If the sum of first p terms of an A.P. is equal to the sum of first q terms then show that the sum of its first $(p + q)$ terms is zero. $(p \neq q)$

13.★ If m times the m^{th} term of an A.P. is eqaul to n times n^{th} term then show that the $(m + n)^{th}$ term of the A.P. is zero.

14. ₹ 1000 is invested at 10 percent simple interest. Check at the end of every year if the total interest amount is in A.P. If this is an A.P. then find interest amount after 20 years. For this complete the following activity.

$$\text{Simple interest} = \frac{P \times R \times N}{100}$$

$$\text{Simple interest after 1 year} = \frac{1000 \times 10 \times 1}{100} = \square$$

$$\text{Simple interest after 2 year} = \frac{1000 \times 10 \times 2}{100} = \square$$

$$\text{Simple interest after 3 year} = \frac{\square \times \square \times \square}{100} = 300$$

According to this the simple interest for 4, 5, 6 years will be 400, $\square$, $\square$ respectively.

From this $d = \square$, and $a = \square$

Amount of simple interest after 20 years

$$t_n = a + (n-1)d$$

$$t_{20} = \square + (20-1)\square$$

$$t_{20} = \square$$

Amount of simple interest after 20 years is = $\square$